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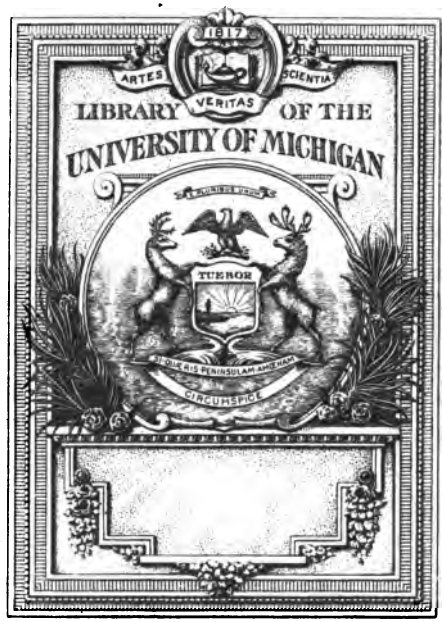

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TREATISE

ON

ALGEBRA,

IN PRACTICE AND THEORY,

IN TWO VOLUMES,

WITH

NOTES AND ILLUSTRATIONS;

CONTAINING

A VARIETY OF PARTICULARS RELATING TO THE DISCOVERIES AND IMPROVEMENTS THAT HAVE BEEN MADE IN THIS BRANCH OF ANALYSIS.

BY JOHN BONNYCASTLE,

PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY, WOOLWICH.

VOL. II.

LONDON:

PRINTED FOR J. JOHNSON AND CO.;

J. NUNN; C. LAW; LONGMAN, HURST, REES, ORME, AND BROWN; CADELL AND DAVIES; B. AND R. CROSBY AND CO.; J. RICHARDSON; SHERWOOD, NEELY, AND JONES; CRADOCK AND JOY; AND G. AND S. ROBINSON.

MDCCCXIII.

Printed by C. WOOD,
Poppin's Court, Fleet Street.

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ERRATA.

Page 2, l. 27, *for* -1, *read* $-$.
8, last line, *for* $\frac{+ab}{+b}$, *read* $\frac{+ab}{a}$.
14, l. 22, *for* $\sqrt{(-a)}$, *read* $\sqrt[2n]{(-a)}$.
31, l. 16, *for* products, *read* product.
34, l. 5, *for* left, *read* right.
59, l. 15, *for* $+6$, *read* -6.
96, l. 11, *for* $\frac{1}{2}a^2$, *read* $\frac{1}{4}b^2$.
123, in the note, *read Resolution des équations numériques.*
237, l. 17, *for* $\frac{6}{4}$, *read* $\frac{4}{4}$.
388, l. 17, *for* length of OA or OB, *read* length of AVB.

ALGEBRA.

LEADING RULES OF THE SCIENCE.

(A) THE definitions, and mode of notation, laid down in the corresponding article of the preceding part of this work, having been, in general, sufficiently explained, it will here only be necessary to observe, that the difficulties commonly experienced by learners, in comprehending the nature of some of the first rules of algebraical computation, arise chiefly from the operations being performed by means of general symbols, instead of by numbers, and the consequent necessity of giving the sign — a more extensive signification than that of barely indicating the subtraction of the quantities, before which it is placed, from those that are marked with the opposite sign +; which circumstances, as far as they relate to the addition of such quantities, may be elucidated as follows:

(B) If the symbol a, for example, in the first case of this rule, be made to represent any quantity, or thing, which is the object of calculation, it is obvious that $5a$ will represent five times that thing, and $7a$ seven times the same thing, whatever may be the denomination or numeral value of a; and consequently, if the quantities $5a$ and $7a$ are to be incorporated, or added together, their

sum will be twelve times the thing denoted by a, or $12a$.

Also, in the other part of this case, which relates to the adding of what are usually called negative quantities, as $-2a$ and $-3a$, it is plain that no other idea can be attached to such expressions, than that of their producing, in any mixed operation, a contrary effect to that of the positive quantities with which they are connected; and consequently, after incorporating them in the same manner as the latter, the sign $-$ must be prefixed to the result; so that if A be greater than a, it is evident that $2(A-a)+3(A-a)$, or $(2A-2a)+(3A-3a)$ will be $=5A-5a$, and therefore the sum of the quantities $-2a$ and $-3a$, when taken in their isolated state, will, by a necessary extension of the rule, be $=-5a$.

Hence, likewise, in the second case of the rule, where the quantities are supposed to be like, but to have unlike signs, the reason of the operation will readily appear, from considering that the addition of algebraic quantities, taken in a general sense, or without any regard to their particular values, means only the uniting of them together, by means of the arithmetical operations denoted by the signs $+$ and $-$; and as these are of contrary, or opposite natures, the less quantity must be taken from the greater, in order to obtain the incorporated mass; and the sign of the greater prefixed to the result. So that if $6a$ is to be added to $4(A-a)$, or to $(4A-4a)$, the result will, evidently, be $4A+2a$; and if $4a$ is to be added to $6(A-a)$, or to $(6A-6a)$, the result will be $6A-2a$; whence,

by making the rule general, as in the former case, the sum of $6a$ and $-4a$ will be $2a$, and that of $4a$ and $-6a$ will be $-2a$.

Also, with respect to the third case, where the quantities are supposed to be unlike, or of different kinds, it is manifest that they can be no otherwise incorporated, or added together, than by means of the signs $+$ and $-$; for if a, for example, be made to represent a pound, and b a shilling, the sum of a and b can be neither $2a$ nor $2b$, that is, neither two pounds nor two shillings, but one pound *plus* one shilling; which must therefore be expressed by $a+b$.

(c) This rule is founded on the consideration, that addition and subtraction are directly opposite to each other, both in their nature and effect, like the signs $+$ and $-$, by which they are denoted; so that to subtract one positive quantity from another, of the same kind, is equivalent to adding the same quantity taken negatively; and to subtract a negative quantity, is equivalent to adding the same quantity taken affirmatively; each of which operations may therefore be performed, as directed in the rule, by changing the sign of the quantity that is to be subtracted, from $+$ to $-$, or from $-$ to $+$, and then proceeding as in addition.

Thus, if $3a$ is to be subtracted from $8a$, the result will be $8a-3a$, which is equal to $5a$; and if $2b-c$ is to be subtracted from $3a$, the result will be $3a-(2b-c)$, which is equal to $3a-2b+c$; for since, in this case, it is the difference between $2b$ and c that is to be taken from $3a$, it is plain, from the quantity that is to be subtracted being less than $2b$ by c, that if the affirmative part $2b$ be only taken

away, too much will have been deducted by the quantity c, and therefore c must be added to the result, in order to render it correct.

(D) The rule for multiplying such algebraic quantities together as are affirmative, is the same as that for the multiplication of numbers, in common arithmetic, except as far as regards the generality of the terms, and the practice made use of, in the former, of reversing the order of the operation.

Thus, if a is to be multiplied by b, or b by a, the product, in this case, is usually indicated, without any particular process, by $a \times b$, or ab; which signifies, that the number represented by a is to be taken as many times as there are units in the number represented by b; or that b is to be taken as often as there are units in a; so that, for instance, if the numeral value of a be 12, and that of b 8, the product ab will be 12×8, or 8×12, which, in either case, is 96 (*a*).

(*a*) Several modern writers on algebra have thought it necessary to prove, that, if a and b be any two unequal numbers, the product of a by b will be the same as the product of b by a: or that

$$ab = ba.$$

And it must be confessed, that the proposition, though undoubtedly true, is not so evident as our general habits, acquired from the practice made use of in multiplying small numbers, would lead us to imagine. In order, therefore, to satisfy the more scrupulous reader on this point, the following demonstration of it is here subjoined.

Let $a \asymp b$, and suppose a to be decomposed into as many numbers b as are contained in it, and put $r =$ the remainder; then

$$a \times b = b(b + b + b + \ldots\ldots + r), \text{ or}$$
$$ab = bb + bb + bb + \ldots + rb.$$

And if any compound quantity, as $b+c+d$&c. is to be multiplied by some simple quantity a, the product, according to the rule, will be found by multiplying each of the parts b, c, d, &c. separately, by a, and then adding the results together, as below:

$$\begin{array}{l} b+c+d+\ \&c. \\ a \\ \hline ab+ac+ad+\ \&c. \end{array}$$

Which process is evidently right, since the sum of the products, arising from multiplying the several parts of any one number by another, is equal to the product of the two entire numbers, as is shown in Euclid's Elements, Book II, prop. 1.

In like manner, if the compound quantity $c+d$, is to be multiplied by some other compound quantity $a+b$, the product, in this case, will be found, by multiplying all the parts of the one by the se-

---

Where it is plain, that the order of all the terms, bb, bb, &c. can be inverted at pleasure.

It therefore only remains to show, that $rb=br$; for which purpose, as $r \rightleftharpoons b$, suppose b to be decomposed into all the numbers r which it contains, as in the first case, and put $r'=$ the remainder; then

$$r \times b = r(r+r+r+\ \ldots\ldots\ +r'),\ \text{or}$$
$$rb = rr+rr+rr+\ \ldots\ldots\ +rr'.$$

Where the order of the terms rr, rr, &c. are obviously convertible, as before.

Hence, by proceeding in this manner, with rr' and all the following products $r'r''$, $r''r'''$, &c. it is plain that we shall, at length, arrive at a decomposition in which the remainder will be either 0 or 1.

And since, agreeably to what is above shown, we can invert,

veral parts of the other, and then adding the results together as before; thus,

$$\begin{array}{c} c+d \\ a+b \\ \hline ac+ad+bc+bd \end{array}$$

Which result must also be just, for reasons similar to those given above; and the same will hold for the multiplication of any polynomial whatever, of this kind, by another.

Again, if B, b, be any two quantities, of which B is the greater, and $B-b$ is to be multiplied by a, it is plain that the product, in this case, must be less than aB, because $B-b$ is less than B; and consequently, when each of the terms of the former are multiplied by a, as above, the result will be

---

in the first of these cases, the order of the factors in all the terms of the product corresponding with ab, it follows that ab itself is likewise convertible into ba.

Also, if n, in the second case, be made to denote the number of accents, that should be placed over the remainder that precedes unity, we shall only have to prove, that $1 \times r^{(n)} = r^{(n)} \times 1$; which being of itself evident, the proposition is true.

Hence, likewise, it may be readily shown, that the order of the factors, in the product of any three numbers abc, can be inverted at pleasure without altering the result.

For since, from the former part of the proposition, we have

$$ab = ba, \quad ac = ca, \text{ and } bc = cb,$$

it follows, by a proper substitution of these equals in the original and succeeding products, that

$$abc = bac = bca = cba = acb = cab.$$

And the same mode of reasoning may be applied to four, five, &c. factors; and so on, by degrees, to any number of factors whatever.

Whence the rule is universally true, in all cases of this kind.

$$\begin{array}{r} \text{B} - b \\ a \\ \hline a\text{B} - ab \end{array}$$

For if it were $a\text{B} + ab$, the product would be greater than $a\text{B}$, which is absurd.

Also, if B be greater than b, and A greater than a, and it is required to multiply $\text{B} - b$ by $\text{A} - a$, the operation, according to the rule, will stand thus:

$$\begin{array}{r} \text{B} - b \\ \text{A} - a \\ \hline \text{AB} - \text{A}b - a\text{B} + ab \end{array}$$

In which case the result must, likewise, be true; for the product of $\text{B} - b$ by A is $\text{A}(\text{B} - b)$, or $\text{AB} - \text{A}b$, and that of $\text{B} - b$ by $-a$, which is to be taken from the former, is $-a(\text{B} - b)$, as has been already shown; whence $\text{B} - b$ being less than B, it is evident that the part which is to be taken away must be less than $a\text{B}$; and consequently, since the first part of this product is $-a\text{B}$, the second part must be $+ab$; for if it were $-ab$, a greater part than $a\text{B}$ would be to be taken from $\text{A}(\text{B} - b)$, which is absurd.

Hence, from what has been here proved, it follows, that like signs, in multiplication, give $+$, and unlike signs $-$, agreeably to what has been observed in the practical rule, Part I: that is,

1. $+a \times +b = +ab$
2. $-a \times +b = -ab$
3. $+a \times -b = -ab$
4. $-a \times -b = +ab$

It is here likewise farther evident, that, although

no direct mode of reasoning can be applied to such operations as $(+a) \times (-b)$, or $(-a) \times (-b)$, when the quantities are isolated, or taken separately; yet, in all cases of actual multiplication, where $-a$ or $-b$ are mixed with real quantities, the rule for the signs of these products has been shown to be true, and therefore ought to be made general (*b*).

(E) The rule of division being merely the converse of that for multiplication, it would be needless to offer any thing, by way of demonstration, respecting the principles on which it is founded, or the truth of the results, obtained in the several cases into which it is divided, as these will be found sufficiently clear from what has been delivered in the last article.

It therefore only remains to show, why, in division, as well as in multiplication, like signs produce $+$, and unlike signs $-$; in order to which it must be observed, that, according to the nature of division, the product of the quotient and divisor ought to be equal to the dividend.

Hence, it is evident, that

1. $\frac{+ab}{+b} = +b$; because $+a$ into $+b$, gives $+ab$.

(*b*) The method here used appears to be the only way by which, from evident principles, we can prove the truth of the above rule; for the mode of illustration, drawn from mercantile transactions, respecting the difference between real effects and debts, is obviously too vague and unscientific to be used in an investigation which is purely analytical; and that derived from multiplying $(a-a)=0$, by $+b$ or $-b$, is founded upon a manifest sophism; for $a-a$ cannot, at once, be put $=0$, and then considered in reasoning as a real quantity.

2. $\frac{-ab}{+a} = -b$; because $+a$ into $-b$, gives $-ab$.

3. $\frac{+ab}{-a} = -b$; because $-a$ into $-b$, gives $+ab$.

4. $\frac{-ab}{-a} = +b$; because $-a$ into $+b$, gives $-ab$.

Which four cases comprehend all the changes that can possibly happen, with regard to the variation of the signs.

We may here, also, farther observe, in addition to what has been said in the practical part of the work, respecting the first four rules of this science, that, even in these simple operations, the great advantage of the algebraic calculus over that of arithmetic, is sufficiently manifest, from the readiness with which it enables us to establish the truth of several general propositions, which, if numbers had been used instead of letters, could only have been inferred from induction.

Thus, from the rules of addition and subtraction only, it appears, that the half difference of any two quantities of the same kind, added to their half sum, is equal to the greater of those quantities; and that half their difference, taken from their half sum, is equal to the less: that is,

$$1.\ a = \frac{a+b}{2} + \frac{a-b}{2}$$

$$2.\ b = \frac{a+b}{2} - \frac{a-b}{2}$$

It is likewise evident, from the rule of multiplication, that the product of the sum and difference of any two quantities is equal to the difference of their squares: that is,

$$(a+b) \times (a-b) = a^2 - b^2$$

And consequently that the difference of the squares of any two quantities, as $a^2 - b^2$, can always be put under the form $(a+b) \times (a-b)$.

Hence, also, it follows, from the division of these quantities, as pointed out by the common rule, that

$$1.\ \frac{a^2-b^2}{a+b} = a - b.$$

$$2.\ \frac{a^2-b^2}{a-b} = a + b.$$

All of which formulæ the learner should carefully retain in his memory.

(F) The operations which are employed for finding the common measure of two or more quantities, as given in Case I, of Algebraic Fractions, are founded upon the principle, that if one quantity be divided by another, and the remainder and divisor have a common measure, this will, also, be a common measure of the divisor and dividend, the truth of which may be shown thus:

Let $q =$ the quotient of D divided by d, and $r =$ the remainder; and suppose c to be the common measure of r and d, or that $r = mc$, and $d = nc$; then, by the nature of division, $D = qd + r = qnc + mc$; and if the first and last of these equals be each divided by c, we shall have $\frac{D}{C} = \frac{qnc + mc}{c} = qn + m =$ a whole quantity; whence c is a measure of D, as well as of d.

And, if $q' =$ the quotient of $\frac{d}{r}$, $r' =$ the remainder, and c' be a common measure of r and r', it may be shown, as before, that this will, also, be a com-

mon measure of d, and consequently of D; and so on, for any other remainder and the preceding divisor; which is conformable to the directions, given in the practical rule, for finding the common measure of two or more quantities.

It may here, also, be farther shown, that the greatest common measure of two quantities will, in no respect, be altered, by multiplying or dividing either of them by any quantity which is not a divisor of the other, or that contains no factor, which is common to them both.

For, let the quantities ab, ac, be taken, of which the common measure is a; then, if ab be multiplied by d, they will become abd and ac; where it is evident, that a is the common measure, as before. And, conversely, if the first of the two quantities abd, ac, be divided by d, they will become ab, ac, where a is still the common measure.

But it will not be the same, if one of the two quantities be multiplied or divided by a quantity which is a divisor of the other, or that has a common factor with it; for if the first of the two quantities ab, ac, be multiplied by c, they will become abc and ac, of which the common divisor is ac, instead of a; and conversely, if the first of the two quantities abc and ac, be divided by c, they will become ab and ac; of which the common divisor is a, instead of ac.

We might here, also, have readily shown the reasons of all the other rules of algebraic fractions; but, as they are founded upon the same principles with those in arithmetic, any particular demonstration of them would be unnecessary.

(G) The rule for involving, or raising quantities to different powers, being similar to that of multiplication, neither requires, nor, indeed, admits of any other proof, than that which arises out of the nature of the signs, and the received method of notation.

Thus, in the involution of monomials, a^2, a^3, &c. . . . a^m, are only abridged modes of representing aa, aaa, ... $aaaa$ &c. taken m times; and therefore any farther illustration of the rule, in this case, would be superfluous.

In like manner, if a^3 is to be squared, or a^2 to be cubed, we shall have, by multiplication, and the nature of powers, $(a \times a \times a) \times (a \times a \times a)$, or $a \times a \times a \times a \times a \times a = a^6$, in the first case; and $(a \times a) \times (a \times a) \times (a \times a)$, or $a \times a \times a \times a \times a \times a$, which is also $= a^6$ in the second case.

And if a^m is to be raised to the nth power, the result will be obtained by finding the continued product of ($a \times a \times a$ &c. to m factors) into ($a \times a \times a$ &c. to n factors) &c. repeated n times, or $a \times a \times a$&c. repeated nm times, which, according to the common method of notation, is $= a^{mn}$.

Also, in compound quantities, $(a+x) \times (a+x)$ denotes that $(a+x)$ is to be taken $(a+x)$ times, which is evidently equal to $(a+x)$ taken a times $+$ $(a+x)$ taken x times; and this, again, is equal to a taken a times $+x$ taken a times $+a$ taken x times $+x$ taken x times; or $= a^2 + 2ax + x^2$. Whence, as above, $(a+x) \times (a+x)$, $(a+x) \times (a+x) \times (a+x)$, . . . $(a+x) \times (a+x) \times (a+x)$&c. taken m times, are represented by $(a+x)^2$, $(a+x)^3$, . . . $(a+x)^m$.

And therefore

$(a+x)^2 = a^2 + 2ax + {}^2,$

$(a+x)^3 = (a^2 + 2ax + x^2) \times (a+x) = a^3 + 3a^2x + 3ax^2 + x^3,$

$(a+x)^m = a^m + ma^{m-1}x$&c. . . . . x^m

Hence, also, as like signs, in multiplication, produce $+$, and unlike signs $-$, it follows, from the nature of involution, that every even power of any positive or negative quantity will be positive, and that every odd power will have the same sign as the quantity itself: thus,

1. $(\pm a)^{2m} = \{(\pm a)^2\}^m = (+a^2)^m = +a^{2m},$
2. $(\pm a)^{2m+1} = (\pm a)^{2m} \times (\pm a) = +a^{2m} \times \pm a = \pm a^{2m+1}.$

Where, if m be made to represent any whole positive number, $2m$ will always be an even number, and $2m+1$ an odd number.

(H) Evolution being directly opposite in its nature to involution, all that it will be here necessary to observe, with respect to this part of the subject, is, that the methods of extracting the square, cube, and other roots of compound algebraical quantities, are similar to those for numbers, having been chiefly derived from observations made on the nature and formation of different powers; and thence, by trial, finding the several terms of the root. Thus, having by involution,

$(a+b+c+d\&c.)^2 = a^2 + (2a+b)b + \{2(a+b)+c\}c + \{2(a+b+c)+d\}d\&c.$

it is easy to perceive how, by the reverse operation, we can obtain the root $a+b+c+d$ &c. according to the method laid down in the practical rule, Part I.

Also, if the same expression be cubed, we shall have

$$(a+b+c+d\&c.)^3 = a^3 + \{3a^2+3ab+b^2\}b + \{3(a+b)^2+3c(a+b)+c^2\}c + \{3(a+b+c)^2+3d(a+b+c)+d^2\}d\&c.$$

From which, by a similar conversion of the formula, the method of finding the root $a+b+c+d\&c.$ as in the common rule, is equally evident.

And in nearly the same manner may the fourth, fifth, or any other numeral root of a compound quantity be determined; but when they are of the form $(a+b)^{\frac{1}{n}}$, or $(a-b)^{\frac{1}{n}}$, the extraction will be more readily performed by means of the binomial theorem, Art. (Y), Part I.

It will, likewise, readily appear, from reverting the formula in the latter part of the last article, that any even root of an affirmative quantity, as $\sqrt{a}$, or $\sqrt[2n]{a}$, may be either positive or negative, except when the proper sign is indicated by the nature of the question; and that every even root of a negative quantity, as $\sqrt{-a}$, or $\sqrt[2]{(-a)}$, is impossible; there being no quantity that, when multiplied by itself an even number of times, will give a negative product.

Also, because every odd power of any positive or negative quantity always preserves the sign of its root, it follows, that every odd root of an affirmative quantity will be positive, and every odd root of a negative quantity negative; so that, in the latter case,

$$\sqrt[3]{-a} = -\sqrt[3]{a}, \text{ and } \sqrt[2n+1]{-a} = -\sqrt[2n+1]{a}.$$

Where it appears that the sign $-$ may be placed either before the radical or under it, as may be thought necessary.

(1) Most of the common rules for the management of surds being of themselves sufficiently evident, it will here only be necessary to notice such of the more difficult cases of this kind, as, from their peculiarity or importance, appear to require a separate demonstration.

In the first place, then, it will be proper to show, by way of lemma to what follows, that

1. If any two numbers, a and b, be each prime to another number, p, their product, ab, will not be divisible by p (c).

For suppose the division to be possible; and because a and b, by hypothesis, are each prime to p, let $a = mp \pm r$, and $b = np \pm r'$; where m and n may be so taken that both r and r' shall be less than p.

Then, since, by multiplication, $ab = mnp^2 \pm mr'p \pm nrp + rr'$, if the first side, ab, of this equation, be divisible by p, the other side must also be divisible by it.

But the first three terms, $mnp^2 \pm mr'p \pm nrp$, of the latter, are evidently divisible by p; whence the last term rr' must also be divisible by p.

And since r, r' are here each less than p, let, in

(c) This proposition is one of the most important in the theory of numbers; as the doctrine of incommensurables, so largely treated of by Euclid, and other writers on Geometry, entirely depends upon it. Thus, in the next theorem, which is merely a simple deduction from the present one, it appears that the roots of all imperfect powers, such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{2}$, $\sqrt[3]{3}$, &c. are inexpressible either in whole numbers or fractions, and can therefore only be expounded by non periodic decimals, that never terminate; on which account they are called irrational, or incommensurable numbers, as they cannot be compared either with integers or fractions.

this case, $p = \mu r - r''$, where μ may be so taken that r'' shall be less than r, and we shall have, by multiplying by r', $r'p = \mu rr' - r'r''$ (*d*).

Hence, because the first side of this equation is divisible by p, the other side must also be divisible by it.

But the first term $\mu rr'$ of the latter is divisible by p, because rr', as above shown, is so; whence the remaining term $r'r''$, which is less than rr', must likewise be divisible by p.

Also, by again taking $p = \nu r' - r'''$, so that r''', as before, may be less than r', and multiplying by r'', we shall have $r''p = \nu r'r'' - r''r'''$; where it will be found, as in the last case, that $r''r'''$, which is less than $r'r''$, must also be divisible by p.

And, by following continually the same process, we shall, at length, necessarily arrive at a number $r^{(n)}r^{(n+1)}$, which is less than p, and consequently not divisible by it.

Whence, if a and b be each prime to p, their product ab will not be divisible by p.

Cor. The proposition will also be found to hold equally true for any number of factors, abc, $abcd$, &c. when a, b, c, &c. are each prime to some other number π.

(*d*) It may here be observed, that r, r', and all the following remainders, in this demonstration, are supposed to be greater than unity, for if either of them, as r, were $= 1$, the proposition, in that case, would be evidently true, since $\frac{rr'}{p}$ would then become $\frac{r'}{p}$, which cannot be a whole number, because r' has been taken less than p.

For since ab, considered as a single number, is prime to π, as has been shown above, and c is also prime to it, by hypothesis; therefore $ab \times c$, or abc, is not divisible by π.

Also, because abc, taken as a single number, is prime to π, and d is likewise prime to it, $abc \times d$, or $abcd$, is not divisible by π; and so on indefinitely.

2. Hence, also, it appears, that if two numbers a and b be prime to each other, the fraction $\frac{a^m}{b^m}$ cannot be reduced to a whole number.

For, by considering the factors in the last proposition as equal to each other, it is evident, that $\frac{a^m}{b}$, or its equal $\frac{a \times a \times a \ldots \text{to } m \text{ terms}}{b}$ can never become a whole number.

Also, by supposing $\frac{a^m}{b^2}$ equal to some whole number q, we shall have, in that case, $\frac{a^m}{b} = bq$, a whole number, which is impossible.

Whence, by proceeding in a similar manner, the same may be shown of all the other powers,

$$\frac{a^m}{b^3}, \frac{a^m}{b^4}, \ldots \ldots \frac{a^m}{b^n}, \frac{a^m}{b^m}.$$

3. From the same principles it likewise follows, that no whole number can have a vulgar fraction for its root.

For, let $\sqrt[m]{n} = \frac{a}{b}$, where $\frac{a}{b}$ is a fraction in its lowest terms, and we shall have, by involution, $\frac{a^m}{b^m} = n$, which has been shown to be impossible.

4. It may, also, be farther proved, that the

square, cube, &c. root of a whole number, cannot be partly rational and partly irrational.

For, if possible, let $\sqrt{a} = x \pm \sqrt{y}$, for the square root.

Then, by squaring both sides, $a = x^2 \pm 2x\sqrt{y} + y$; and, by transposition, $\pm 2x\sqrt{y} = a - x^2 - y$.

Whence, by division, $\pm\sqrt{y} = \frac{a - x^2 - y}{2x} =$ a rational quantity, which is contrary to the supposition.

Also, let, if possible, $\sqrt[3]{a} = x \pm \sqrt{y}$, for the cube root.

Then, by cubing both sides of the equation, we have $a = x^3 \pm 3x^2\sqrt{y} + 3xy \pm y\sqrt{y}$.

And, by transposition, $\pm(3x^2 + y)\sqrt{y} = a - x^3 - 3xy$.

Whence, $\pm\sqrt{y} = \frac{a - x^3 - 3xy}{3x^2 + y} =$ a rational quantity; which is also contrary to the supposition.

Again, let, if possible, $\sqrt[3]{a} = x \pm \sqrt[3]{y}$, or $\sqrt[3]{a} - x = \pm\sqrt[3]{y}$.

Then, by involution and transposition, we shall have $3x^2\sqrt[3]{a} - 3x\sqrt[3]{a^2} = x^3 - a \pm y$.

And, by multiplying each side of this equation by $\sqrt[3]{a}$, $3x^2\sqrt[3]{a^2} - 3ax = (x^3 - a \pm y)\sqrt[3]{a}$.

Whence, multiplying the first of these equations by x, and then adding it to the second, we shall have $3x^3\sqrt[3]{a} - 3ax = x^4 - ax \pm xy + (x^3 - a \pm y)\sqrt[3]{a}$.

Where it will be readily found, by transposition and division, that

$$\sqrt[3]{a} = \frac{x^4 + 2ax \pm xy}{2x^3 + a \mp y}.$$

Which, being a rational quantity, the supposition, in this case, is likewise impossible.

And in nearly the same way may the proposition be proved, for the 4th, and other higher roots.

5. In any equation of the form $x \pm \sqrt{y} = a \pm \sqrt{b}$, the rational parts on each side are equal, and also the irrational parts.

For, if x be not equal to a, let it be equal to $a \pm z$.

Then $a \pm z \pm \sqrt{y} = a \pm \sqrt{b}$, or $\pm \sqrt{b} = \pm z \pm \sqrt{y}$.

That is, $\sqrt{b}$ is partly rational and partly irrational, which has been shown to be impossible.

Whence $x = a$, and consequently $\sqrt{y} = \sqrt{b}$.

The same is also true for the equation $x \pm \sqrt[3]{y} = a \pm \sqrt[3]{b}$.

For let $x = a \pm z$, as before, then $a \pm z \pm \sqrt[3]{y} = a \pm \sqrt[3]{b}$, or $\pm \sqrt[3]{b} = \pm z \pm \sqrt[3]{y}$, which, by the last proposition, is also impossible; and so on for any other roots.

6. If $\sqrt{(a + \sqrt{b})} = x + \sqrt{y}$, then will $\sqrt{(a - \sqrt{b})} = x - \sqrt{y}$; and if $\sqrt[3]{(a + \sqrt{b})} = x + \sqrt{y}$, then $\sqrt[3]{(a - \sqrt{b})} = x - \sqrt{y}$; &c.

For since, by involution, $a + \sqrt{b} = x^2 + 2x\sqrt{y} + y$, we shall have, by the last proposition,

$$a = x^2 + y$$
$$\sqrt{b} = 2x\sqrt{y}$$

Whence $a - \sqrt{b} = x^2 - 2x\sqrt{y} + y$

And because $x^2 - 2x\sqrt{y} + y = (x - \sqrt{y})^2$; therefore, by evolution, $\sqrt{(a - \sqrt{b})} = x - \sqrt{y}$.

Again, for the cube root; since $a + \sqrt{b} = x^3 + 3x^2\sqrt{y} + 3xy + y\sqrt{y}$, there will arise, as before,

$$a = x^3 + 3xy$$
$$\sqrt{b} = 3x^2\sqrt{y} + y\sqrt{y}$$

Or $a - \sqrt{b} = x^3 - 3x^2\sqrt{y} + 3xy - y\sqrt{y}$

Whence, because $x^3 - 3x^2\sqrt{y} + 3xy - y\sqrt{y} = (x - \sqrt{y})^3$, we shall have $\sqrt[3]{(a - \sqrt{b})} = x - \sqrt{y}$.

And, by expanding $(x + \sqrt{y})^m$, by the binomial theorem, it may be proved, in a similar manner, that if $(a + \sqrt{b})^{\frac{1}{n}} = x + \sqrt{y}$, then $(a - \sqrt{b})^{\frac{1}{n}} = x - \sqrt{y}$.

7. In any expression of the quadratic form $px^{2m} + qx^m + r$, if $4pr = q^2$, that expression will be a complete square.

For, by dividing each member of the equation by $4p$, we shall have

$$r = \frac{q^2}{4p}.$$

And if this value be substituted for r in the original expression, it will become

$$px^{2m} + qx^m + \frac{q^2}{4p}.$$

Which is evidently a square, being equal to

$$(p^{\frac{1}{2}}x^m + \frac{q}{2p^{\frac{1}{2}}})^2.$$

8. It may here also be shown, how the square root of a binomial or residual surd, may sometimes be made equal to the sum, or difference of two other surds.

Thus, let $\sqrt{(a + \sqrt{b})} = x + y$; then $\sqrt{(a - \sqrt{b})} = x - y$, as above.

And, by squaring each of these equations,

$$a + \sqrt{b} = x^2 + 2xy + y^2$$
$$a - \sqrt{b} = x^2 - 2xy + y^2$$

Whence $2a = 2x^2 + 2y^2$; or $a = x^2 + y^2$

Also, by multiplying the like sides of the same two equations, $\sqrt{(a + \sqrt{b})} \times \sqrt{(a - \sqrt{b})} = x^2 - y^2$; or $\sqrt{(a^2 - b)} = x^2 - y^2$.

Wherefore, by addition and subtraction, $a+\sqrt{(a^2-b)}=2x^2$, and $a-\sqrt{(a^2-b)}=2y^2$; or $x=\sqrt{\{\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)}\}}$, and $y=\sqrt{\{\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)}\}}$.

Whence

$$\sqrt{(a+\sqrt{b})}=\sqrt{\{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2-b)}\}}+\sqrt{\{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2-b)}\}}$$
$$\sqrt{(a-\sqrt{b})}=\sqrt{\{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2-b)}\}}-\sqrt{\{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2-b)}\}}$$

And, if b be negative instead of affirmative, the two formulæ, in that case, will become

$$\sqrt{(a+\sqrt{-b})}=\sqrt{\{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2+b)}\}}+\sqrt{\{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2+b)}\}}$$
$$\sqrt{(a-\sqrt{-b})}=\sqrt{\{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2+b)}\}}-\sqrt{\{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2+b)}\}}$$

agreeably to what has been observed in the practical rule, given in the note to Case 11, Art. (1).

9. The cube root of a binomial or residual surd may also be obtained, in some particular cases, as follows:

Let $\sqrt[3]{(a+\sqrt{b})}=x+\sqrt{y}$, or, in order to render the form more general, $\sqrt[3]{(a+\sqrt{b})}=(x+\sqrt{y})\sqrt[3]{c}$.

Then, by cubing each side of this expression, we shall have

$$a+\sqrt{b}=c(x^3+3x^2\sqrt{y}+3xy+y\sqrt{y}).$$

And, by comparing the rational and irrational parts of the formula so obtained with each other,

$$a=c(x^3+3xy),\ \sqrt{b}=c(3x^2+y)\sqrt{y}.$$

Or, by squaring each side of these equations,

$$a^2=c^2(x^6+6x^4y+9x^2y^2)$$
$$b=c^2(9x^4y+6x^2y^2+y^3)$$

Whence, also, by subtraction and division, we shall have

$$\frac{a^2-b}{c^2}=x^6-3x^4y+3x^2y^2-y^3=(x^2-y)^3.$$

And consequently, by extracting the cube root of each side

$$x^2 - y = \left(\frac{a^2 - b}{c^2}\right)^{\frac{1}{3}} = \frac{1}{c}\sqrt[3]{\{(a^2 - b)c\}}.$$

Where let c be so taken that $\sqrt[3]{\{(a^2 - b)c\}}$ shall be a perfect cube, and put $\frac{1}{c}\sqrt[3]{\{(a^2 - b)c\}} = c$.

Then, since $x^2 - y = c$, and consequently $y = x^2 - c$, if this value be substituted in the equation $a = c(x^3 + 3xy)$, it will become

$$4cx^3 - 3ccx = a.$$

In which case the value of x will be rational, when the extraction in question is possible; and consequently may be readily found by inspection.

Let, therefore, this value of x, so determined, be put $= r$; then, because $y = x^2 - c$, we shall have

$$\sqrt[3]{(a + \sqrt{b})} = r + \sqrt{(r^2 - c)}$$
$$\sqrt[3]{(a - \sqrt{b})} = r - \sqrt{(r^2 - c)}$$

agreeably to the rule given in the note to Art. (1), Case 12, Part 1; where c is taken as unity.

10. And by following a process similar to the last, any root whatever of a binomial or residual surd, may be determined, in such cases as admit of this kind of extraction.

Thus, let $\sqrt[n]{(a + \sqrt{b})} = (x + \sqrt{y})\sqrt[n]{c}$; and we shall have, by involution,

$$a + \sqrt{b} = c(x + \sqrt{y})^n.$$

An equation which, by expanding the right hand member, and comparing the rational and irrational parts, gives

$$a = c\left(x^n + \frac{n(n-1)}{2}x^{n-2}y + \frac{n(n-1)(n-2)(n-3)}{2.3.4}x^{n-4}y^2 \&c.\right)$$

$$\sqrt{b} = c\left(nx^{n-1}\sqrt{y} + \frac{n(n-1)(n-2)}{2.3}x^{n-3}y\sqrt{y} \quad . \quad . \quad . \quad . \quad . \quad \&c.\right)$$

Or, which is the same thing, under a different form,

$$a = \tfrac{1}{2}c\{(x+\sqrt{y})^n + (x-\sqrt{y})^n\}$$
$$\sqrt{b} = \tfrac{1}{2}c\{(x+\sqrt{y})^n - (x-\sqrt{y})^n\}$$

Whence, by squaring each of these equations, and subtracting the latter from the former, we shall have

$$a^2 - b = \tfrac{1}{4}c^2\{(x+\sqrt{y})^{2n} + 2(x^2-y)^n + (x-\sqrt{y})^{2n}\} -$$
$$\{(x+\sqrt{y})^{2n} - 2(x^2-y)^n + (x-\sqrt{y})^{2n}\}$$

Or, leaving out the terms that destroy each other, and then multiplying by $\frac{1}{4}$,

$$a^2 - b = c^2(x^2-y)^n, \text{ or } x^2 - y = \left(\frac{a^2-b}{c^2}\right)^{\frac{1}{n}}$$

Where, let c be so taken that $\frac{a^2-b}{c^2}$ shall be a complete power of the degree n, and put $\left(\frac{a^2-b}{c^2}\right)^{\frac{1}{n}} = c$

Then, since $x^2 - y = c$, and consequently $y = x^2 - c$, if this value be substituted for y, in the equation

$$c\left(x^n + \frac{n(n-1)}{2}x^{n-2}y + \frac{n(n-1)(n-2)(n-3)}{2.3.4}x^{n-4}y^2 \&c.\right. = a,$$

we shall obtain an equation in which the value of x will be rational, when the extraction required is possible (*e*).

(K) The signification and use of imaginary quantities, or the symbolical expressions commonly so called, having been sufficiently explained in the former part of the work, it will here only be neces-

(*e*) Newton, Maclaurin, and other writers on Algebra, have given rules somewhat different from this, for extracting the nth root of a binomial surd; but they are all more or less tentative, and, except for the square and cube roots, are of little use; as it is but seldom that any method of this kind will be found to succeed when applied to the higher extractions.

sary to give some account of that part of the practical rule which relates to the method of determining their roots and products, as the rest of the operations are either deducible from these, or performed in the same way as for real quantities.

In the first place, then, it may be observed, that since $(+a) \times (+a)$ and $(-a) \times (-a)$ are each, by the established rules of multiplication, $= +a^2$, it follows, reversedly, that when a^2 is considered abstractedly, or without any regard to its generation, the $\sqrt{a^2}$ may be either $+a$ or $-a$, there being nothing in the nature of the quantity so taken, to denote from which of these two expressions it was derived.

But this ambiguity, which, in the abovementioned case, arises from our being unacquainted with the origin of the quantity whose root is to be extracted, will not take place when the sign of the quantity from which it was produced is known; as there can, then, be only one root, which must evidently be taken in plus or minus, according to the state it existed in before it was involved.

Thus $\sqrt{\{(+a) \times (+a)\}}$, or $\sqrt{\{(+a)^2\}}$, cannot be of the ambiguous form $\pm a$, as it would have been if a^2 had been unconditionally assumed, but is simply a, and, for a like reason, $\sqrt{\{(-a) \times (-a)\}}$, or $\sqrt{(-a)^2}$ is $= -a$, and not $\pm a$; since the value of the equivalent expression $\sqrt{a^2}$, in both these cases, is determined, from the circumstance of its being previously known how a^2 is derived (*f*).

(*f*) It may here be observed, that the restriction abovementioned is sometimes found to take place in the solution of nume-

Hence, if it were required to find the product of the two imaginary factors $\sqrt{-1} \times \sqrt{-1}$, or, which is the same thing, the root of the equivalent expression $\sqrt{(-1)^2}$, it is plain, from what has been above said, that this is $= -1$, and can never be of any other form.

Also, if the factors, in this case, be both negative, the result will be the same as before; since $-\sqrt{-1} \times -\sqrt{-1} = \sqrt{(-1)^2} = -1$; but if one of the factors be positive and the other negative, we shall have $+\sqrt{-1} \times -\sqrt{-1} = -\sqrt{(-1)^2} = 1$.

And if the quantities that are to be involved, or multiplied, are of the form $\sqrt{-a}$ and $\sqrt{-b}$, it will only be necessary to write $a^{\frac{1}{2}}\sqrt{-1}$ for $\sqrt{-a}$, and $b^{\frac{1}{2}}\sqrt{-1}$ for $\sqrt{-b}$, and we shall have $\sqrt{(-a)^2} = (a^{\frac{1}{2}}\sqrt{-1})^2 = -a$, and $\sqrt{-a} \times \sqrt{-b} = a^{\frac{1}{2}}\sqrt{-1} \times b^{\frac{1}{2}}\sqrt{-1} = -\sqrt{ab}$.

So that, by always separating the quantities into such factors that -1 only shall remain under the radical, all the rules in the practical article, for involving and multiplying imaginaries, may be rea-

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ral equations: thus, for example, if it were required to find the value of x in the equation $\sqrt{(12+x)} = 6 + \sqrt{x}$, we shall have, by operating in the usual manner, $x = 4$; but the square root of 4, must, in this case, be -2 and not $+2$, as the latter supposition will not answer the conditions of the equation, on account of $\sqrt{x}$ having been first found $= -2$, and then $x = 4$; whence, in retracing the steps of the operation, it is evident, that the manner, in which this 4 was generated, does not admit of its having two roots, or being denoted by the ambiguous sign $\pm$, as it would have been, if taken without any regard to its origin; and the same may be shown of several other equations of this kind.

dily derived from the examples here given; and the reverse of these must be used for division.

Wherefore, as the principle, last mentioned, holds in all cases that can be proposed, it is evident that there is only, in Algebra, the single imaginary quantity $\sqrt{-1}$. And it has been farther shown, by D'Alembert, in the Berlin Memoirs, and since by other writers, that every imaginary expression, however complicated, can be reduced to the form $a+b\sqrt{-1}$, or $a-b\sqrt{-1}$ (g).

ARITHMETICAL PROPORTION AND PROGRESSION.

(L) Most of the foreign writers on Algebra, at present, treat of arithmetical proportion and progression under the denomination of equi-differences, which they consider, not without reason, as a more appropriate appellation than the former, as the term arithmetical conveys no idea of the nature of the subject to which it is applied.

They also represent the relations of these quantities under the form of an equation instead of by points, as is usually done; so that if a, b, c, d, taken

(g) Some modern writers on Algebra have decried the use of all expressions of this kind, representing them as perfectly absurd, and consequently derogatory to the science into which they have been introduced; but as these symbols form one of the constituent parts of analysis, and have constantly been found, when subjected to the established rules of operation, to afford results, which, in all cases, where a comparison can be made, agree with those obtained from the most rigid geometrical principles, they will always be regarded as useful instruments of calculation; particularly as they furnish the means of obtaining several elegant theorems, both in Trigonometry and Physical Astronomy, which could not be easily derived in any other way.

in the order in which they stand, be four quantities in arithmetical proportion, this relation will be expressed, according to the method above mentioned, by

$$a-b=c-d,$$

where it is evident that all the properties of this kind of proportion, can be readily obtained by the mere transposition of the terms of the equation.

Thus, by carrying b to the right hand side of the equation, and d to the left; or, which is the same thing, by adding $b+d$ to each of the two members, we shall have

$$a+d=b+c.$$

From which it appears, as in the common rule, that the sum of the two extremes is equal to the sum of the two means.

And if the third term, in this case, be the same as the second, or $c=b$, the equi-difference is said to be continued, and we have

$$a+d=2b\,;\ \text{or}\ b=\tfrac{1}{2}(a+d)\,;$$

Thus, among others of a similar kind, that might be mentioned, the following well known formulæ,

$$\text{Cos. } x=\frac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2},\ \text{sin. } x=\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{2\sqrt{-1}}$$

where x is any arc of a circle, and e the number whose hyperbolic logarithm is 1, are considered by Lagrange, in his *Leçons sur le Calcul des Fonctions*, p. 114, as one of the finest analytical discoveries of the 17th century.

Professor Playfair, also, in a paper published in the Philosophical Transactions, for 1778, has given several examples in which these imaginary symbols are introduced to advantage; and has observed, "that many researches concerning this kind of analysis, which, in themselves, might be deemed ab-

where it is evident, that the sum of the extremes is double the mean; or the mean equal to half the sum of the extremes.

In like manner, by subtracting $c+d$, and $a+c$, respectively, from each side of the equation $a+d=b+c$, we shall have

$$a-c=b-d,$$
$$d-c=b-a,$$

which shows that the antecedents a and c can be put in the places of the consequents b and d; or, reciprocally, b and d in the places of a and c.

Also, from the equality first taken, $a-b=c-d$, there will result, by addition,

$$(a+m)-(b+n)=(c+m)-(d+n),$$

where it appears that the proportion is not altered, by augmenting the antecedents a and c by the same quantity m, and the consequents b and d by another quantity n.

In short, every operation, by way of addition, subtraction, multiplication, and division, made upon each member of the equation $a-b=c-d$, gives a new property of this kind of proportion, without changing its nature.

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surd, are yet not destitute of utility. John Bernouilli, for instance, has shown, that if the radius of a circle be 1, the circumference will be $=\frac{4 \log.\sqrt{-1}}{\sqrt{-1}}$; which expression, it is true, when considered as a quadrature of the circle, is wholly insignificant, and would deservedly pass for an abuse of calculation; but at the same time, we learn from it, that, if in any equation, the quantity $\frac{4 \log \sqrt{-1}}{\sqrt{-1}}$ enters, it may be made to disappear by the substitution of a circular arc."

ARITHMETICAL PROGRESSION.

THE same principles are also equally applicable to any continued set of equi-differences, of the form

$$a-b=b-c=c-d=d-e\&c.$$

which denote the relations of a series of terms in what has been usually called arithmetical progression.

But these relations will be more commodiously shown, by taking the quantities a, b, c, d, &c. so, that each of them shall be greater or less than that which precedes it by δ; in which case the terms of the series will become

$$a,\ a\pm\delta,\ a\pm 2\delta,\ a\pm 3\delta,\ a\pm 4\delta,\ \&c.$$

Where, if l be put for that term in the progression whose rank is n, its value, according to the law here pointed out, will, evidently, be

$$l=a\pm(n-1)\delta,$$

which expression is usually called the general term of the series; because, if 1, 2, 3, 4, &c. be successively substituted for n, the results will give the rest of the terms.

Also, if s be put equal to the sum of any number of terms of this progression, we shall have

$$s=a+(a\pm\delta)+(a\pm 2\delta)+(a\pm 3\delta)+\ .\ .\ .\ +\{a\pm(n-3)\delta\}+\{a+(n-2)\delta\}+\{a\pm(n-1)\delta\}.$$

And reversing the order of the terms of the series,

$$s=\{a\pm(n-1)\delta\}+\{a\pm(n-2)\delta\}+\{a\pm(n-3)\delta\}+\ .\ .\ .\ +(a\pm 3\delta)+(a\pm 2\delta)+(a\pm\delta).$$

Whence, by adding the corresponding terms of these two equations together, there will arise

$$2s = \{2a \pm (n-1)\delta\} + \{2a \pm (n-1)\delta\} + \{2a \pm (n-1)\delta\} \text{\&c. to } n \text{ terms.}$$

And consequently, as all the terms of this last series are equal to each other, we shall have

$$s = \frac{n}{2}\{2a \pm (n-1)\delta\}.$$

Or, by substituting l for $a \pm (n-1)\delta$, as found above, this expression will become

$$s = \frac{n}{2}(a + l).$$

From which two equations, if any three of the five quantities, a, δ, n, l, s, be given, the rest may be found.

GEOMETRICAL PROPORTION AND PROGRESSION.

(M) THE same reasons that induced the writers mentioned in the last article to give the name of equi-differences to arithmetical proportionals, also led them to apply that of equi-quotients to geometrical proportionals, and to express their relations in a similar way, by means of equations(*h*).

(*h*) This method of representing proportions by means of equations, which has hitherto been chiefly employed upon the continent, is of great use in rendering it unnecessary to burden the memory with a multiplicity of Latin names, which are frequently used to denote the transformations that such quantities may be made to undergo, by changing the order, or altering the values of their terms; and, except that the appellations of equi-differences and equi-quotients are not so simple or classical as could be wished, the notions conveyed by these new definitions are far

Thus, if there be taken any four proportionals, a, b, c, d, which it has been usual to express by means of points, as below,

$$a : b :: c : d,$$

this relation, according to the method above mentioned, will be denoted by the equation

$$\frac{a}{b}=\frac{c}{d}.$$

Where the equal ratios are represented by fractions, the numerators of which are the antecedents, and the denominators the consequents.

Hence, if each of the two members of this equation be multiplied by bd, there will arise

$$ad=bc.$$

From which it appears, as in the common rule, that the products of the two extremes of any four proportionals is equal to that of the means.

And if the third term c, in this case, be the same as the second, or $c=b$, the proportion is said to be continued, and we have

$$ad=b^2, \text{ or } b=\sqrt{ad}.$$

Where it is evident, that the product of the extremes of three proportionals is equal to the square of the mean; or, that the mean is equal to the square root of the product of the two extremes.

Also, if each member of the equation $ad=bc$ be

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more distinct, and conformable to the nature of the subject, than those arising out of the former denominations. Besides which, it may be observed, that the term proportion should only be applied to such quantities as have equal ratios, as those that are formed from addition or subtraction have a totally different relation to each other.

successively divided by bd, dc, ac, &c. the results will give

$\frac{a}{b}=\frac{c}{d}$	Or the proportions	$a : b :: c : d$
$\frac{a}{c}=\frac{b}{d}$		$a : c :: b : d$
$\frac{b}{a}=\frac{d}{c}$		$b : a :: d : c$
&c.		&c.

So that, by following this method, we can easily obtain all the transpositions of the terms of the proportion, that can be made to agree with the equation

$$ad=bc.$$

In like manner, from the equality $\frac{a}{b}=\frac{c}{d}$, there will result, by multiplication, the following equivalent forms:

$$\frac{ma}{mb}=\frac{nc}{nd}\ ;\ \frac{ma}{nb}=\frac{mc}{nd}.$$

Which being converted into proportions become

$$ma : mb :: nc : nd$$
$$ma : nb :: mc : nd.$$

And, by taking any powers, or roots, of the different sides of the same equation, we have

$$\frac{a^m}{b^m}=\frac{c^m}{d^m}.$$

Or, putting the terms in the form of a proportion,

$$a^m : b^m :: c^m . d^m.$$

In which cases m and n may be any whole or fractional numbers whatever.

Again, if there be taken the several equations

$\frac{a}{b}=\frac{c}{d}$	Which correspond with the proportions	$a : b :: c : d$
$\frac{e}{f}=\frac{g}{h}$		$e : f :: g : h$
$\frac{i}{k}=\frac{l}{m}$		$i : k :: l : m$
&c.		&c.

we shall have, by multiplying their like terms,

$$\frac{a \times e \times i\&c.}{b \times f \times k\&c.}=\frac{c \times g \times l\&c.}{d \times h \times m\&c.}.$$

Or, putting the expression into the form of a proportion,

$$aei\&c. : bfk\&c. :: cgl\&c. : dhm\&c.$$

Also, taking $\frac{a}{b}=\frac{c}{d}$, as before, we shall have, by multiplication, $\frac{ma}{nb}=\frac{mc}{nd}$; and by augmenting or diminishing each side of the equation by 1

$$\frac{ma}{nb}\pm 1=\frac{mc}{nd}\pm 1; \text{ or } \frac{ma \pm nb}{nb}=\frac{mc \pm nd}{nd}.$$

Which, being expressed in the form of a proportion, gives

$$ma \pm nb : nb :: mc \pm nd : nd; \text{ or}$$
$$ma \pm nb : mc \pm nd :: nb : nd.$$

And if $\frac{a}{b}=\frac{c}{d}$, be put, by a similar multiplication of its terms, under the form $\frac{pa}{qb}=\frac{pc}{qd}$, and then augmented or diminished by 1, as in the last case, there will arise

$$pa \pm qb : pc \pm qd :: qb : qd.$$

Whence, dividing each of the antecedents of

these two analogies by their consequents, the result will give

$$\frac{ma \pm nb}{mc \pm nd} = \frac{nb}{nd} = \frac{b}{d}; \text{ and } \frac{pa \pm qb}{pc \pm qd} = \frac{qb}{qd} = \frac{b}{d}.$$

And consequently, as the two left hand members of these expressions are each $=\frac{b}{d}$, we shall have

$$\frac{ma \pm nb}{mc \pm nd} = \frac{pa \pm qb}{pc \pm qd}.$$

Or, by converting the corresponding terms of this equation into a poportion

$$ma \pm nb : pa \pm qb :: mc \pm nd : pc \pm qd.$$

Also, because the common equation $\frac{a}{b} = \frac{c}{d}$ gives $\frac{a}{c} = \frac{b}{d}$, if the latter be put under the equivalent forms $\frac{ma}{nc} = \frac{mb}{nd}$, and $\frac{pa}{qc} = \frac{pb}{qd}$, we shall readily obtain, by a similar process,

$$ma \pm nc : pa \pm qc :: mb \pm nd : pb \pm qd.$$

Which two analogies may be considered as general formulæ for changing the terms of the proportion $a : b :: c : d$, without altering its nature.

Thus, by supposing m, n, p, q, to be each $=1$, and taking the antecedents with the superior signs, and the consequents with the inferior, we have

$$a + b : a - b :: c + d : c - d,$$
$$a + c : a - c :: b + d : b - d.$$

Which forms, together with several of those above given, are the usual transformations of the common analogy pointed out in the practical article, Part I.

In like manner, by taking m and n each $=1$, $p=b$, and $q=0$, there will arise, by transposition, and rejecting b in the two consequents,

$$a \pm b : a :: c \pm d : c,$$
$$a \pm c : a :: b \pm d : b.$$

Each of which proportions may be easily verified by making the product of the extremes equal to that of the means, and observing that $ad=bc$.

Lastly, taking any number of equations of the form before used, for expressing proportions, as

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}=\&c.$$

which, according to the common method, are called a series of equal ratios, and are usually denoted by

$$a : b :: c : d :: e : f :: g : h :: \&c.$$

we shall necessarily have, from the fractions being all equal,

$$\frac{a}{b}=q,\ \frac{c}{d}=q,\ \frac{e}{f}=q,\ \frac{g}{h}=q, \&c.$$

And by multiplying q by each of the denominators

$$a=bq;\ c=dq,\ e=fq,\ g=hq,\ \&c.$$

Whence, equating the sum of all the terms on the left hand side of these equations, with those on the right, we have

$$a+c+e+g+\&c.=(b+d+f+h+\&c.)q.$$

And consequently, by division, and the properties of proportionals before shown,

$$\frac{a+c+e+g+\&c.}{b+d+f+h+\&c}=\frac{a}{b}=\frac{a+c}{b+d}=\frac{a+c+e}{b+d+f}=\&c.$$

Which result shows, that, in a series of equal

ratios, the sum of any number of the antecedents is to that of their consequents, as one, or more of the antecedents, is to one, or the same number, of consequents.

Having thus treated of proportion, we may now proceed to progression; which consists of a series of terms so formed, that the quotient arising from dividing either of them by that which precedes or follows it is always the same.

Thus, designing the first term of such a series by a, and the common factor or multiplier by q, the several terms of the progression, taken in order, will be

$$a,\ aq^1,\ aq^2,\ aq^3,\ aq^4,\ aq^5,\ \&c.$$

Where, supposing l to be the general term of the series, or that whose rank is n, its value will evidently be

$$l = aq^{n-1}.$$

Also, if s be made to denote the sum of n terms of the series, including the first, we shall have

$$s = a + aq + aq^2 + aq^3 + \ldots\ldots aq^{n-2} + aq^{n-1}.$$

And multiplying each side of this equation by q, the result will give

$$qs = aq + aq^2 + aq^3 + aq^4 + \ldots\ldots aq^{n-1} + aq^n.$$

Whence, subtracting the first of these equations from the second, and observing that all the terms except a and aq^n destroy each other, we shall have

$$qs - s = aq^n - a;\ \text{or}\ s = a\left(\frac{q^n - 1}{q - 1}\right).$$

Or, substituting l for the last term aq^{n-1}, as above found, this expression will become

$$s = \frac{ql - a}{q - 1}.$$

From which two equations, if any three of the quantities a, q, n, l, s, be given, the rest may be found.

When the common factor q, in the above series, is a whole number, the terms a, aq, aq^2, aq^3, . . . aq^{n-1}, form an increasing progression; in which case n may be so taken, that the value of s shall be greater than any assignable quantity.

But if q be a proper fraction, as $\frac{1}{r}$, the series will then be a decreasing one, and the above expression, by substituting this value for q, will become

$$s = \frac{\frac{a}{r^n} - a}{\frac{1}{r} - 1} \quad \text{or} \quad \frac{ar(1 - \frac{1}{r^n})}{r - 1}.$$

Where it is plain, that the term $\frac{1}{r^n}$ will be indefinitely small when n is indefinitely great; and consequently, by prolonging the series, s may be made to differ from $\frac{ar}{r-1}$ by less than any quantity that can be named.

Whence, by supposing the series to be of the form

$$a + \frac{a}{r} + \frac{a}{r^2} + \frac{a}{r^3} + \frac{a}{r^4} + \frac{a}{r^5} \text{\&c.}$$

and to be continued *ad infinitum*, or without end, we shall, in that case, have

$$s = \frac{ar}{r-1}.$$

Which is what some writers call the radix, and others the limit of the series; as being of such a value, that the sum of no number of its terms can ever exceed it, and yet may be made to approach nearer to it than by any given difference.

DOCTRINE OF EQUATIONS.

(N) THE doctrine of equations being that branch of algebra upon which the solutions of all kinds of analytical problems ultimately depend, I shall here lay down such of the theoretical principles, relating to this part of the science, as appear to be most necessary for the elucidation of the following articles, leaving the rest to be treated of as they occur.

In the first place, then, it may be observed, that the term root, as applied to equations, has a more extensive signification than that which it commonly bears in arithmetic; being here used to denote such a number, or quantity, whether real or imaginary, as, when substituted for the unknown quantity, will make both sides of the equation vanish, or become equal to each other.

1. This being premised, it may now be readily shown, that if a be a root of any equation,

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \ldots + Tx + V = 0 \text{ (i)},$$

(i) The highest term of the equation, in every case of this kind, is always supposed to be positive; and if it has a coefficient prefixed to it, this must be taken away, by dividing all the rest of the terms by it; when it will be reduced to the form of that above given; which is the state that every equation is supposed to be taken in, throughout the following articles.

It is also to be observed, that the coefficients A, B, C, &c. of the proposed equation, as well as the root a, may be taken either in + or in −; and that the proposition will be equally true whether the equation be complete or has some of its terms wanting.

the left hand member of that equation will be exactly divisible by $x-a$.

For, substituting a for x, agreeably to the above definition, we shall necessarily have

$$a^m + Aa^{m-1} + Ba^{m-2} + Ca^{m-3} + \ldots + Ta + V = 0.$$

And consequently, by taking each of the terms, except the last, to the other side of the equation

$$V = -a^m - Aa^{m-1} - Ba^{m-2} - Ca^{m-3} - \ldots - Ta.$$

Whence, if this expression be substituted for V in the first equation, we shall have

$$\left.\begin{array}{l} x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \ldots + Tx \\ -a^m - Aa^{m-1} - Ba^{m-2} - Ca^{m-3} - \ldots - Ta \end{array}\right| = 0.$$

Or, by uniting the corresponding terms, and placing them all in a line,

$$(x^m - a^m) + A(x^{m-1} - a^{m-1}) + B(x^{m-2} - a^{m-2}) + \ldots$$
$$+ T(x-a) = 0.$$

Where, since m is a whole positive number, each of the quantities $(x^m - a^m)$, $(x^{m-1} - a^{m-1})$, $(x^{m-2} - a^{m-2})$, &c. are, by Art. (E), Part I, divisible by $x-a$.

And, therefore, the whole polynomial

$$(x^m - a^m) + A(x^{m-1} - a^{m-1}) + B(x^{m-2} - a^{m-2}) \text{ \&c.}$$

which is equivalent to the first member of the proposed equation, is also divisible by $x-a$; as was to be shown.

But if a be a quantity greater or less than the root, this conclusion will not take place, because, in that case, we shall not have

$$V = -a^m - Aa^{m-1} - Ba^{m-2} - Ca^{m-3} - \ldots - Ta.$$

Which is an equality obviously essential to the division in question.

2. Supposing any proposed equation to have one root, or value of the unknown quantity, it can then be shown, that it will have as many roots as there are units in the index of its highest term, and no more (*k*).

For, omitting simple equations, which have evidently only one root, let us take, in the first place, the general quadratic equation

$$x^2 + px = q.$$

Then if there be any quantity a, which, when substituted for x, will satisfy the conditions of the question, there is, also, some other quantity, or value of x, that will equally satisfy these conditions.

For, putting a in the place of x, we shall have, according to the definition above given,

$$a^2 + pa = q.$$

And consequently, if this value be substituted for q, in the proposed equation, there will arise

$$x^2 + px = a^2 + pa, \text{ or}$$
$$x^2 - a^2 = -p(x - a).$$

And therefore, by dividing this last expression by $x - a$, we shall have

$$x + a = -p, \text{ or}$$
$$x = -(a + p).$$

Whence it appears, that if a be one of the values

(*k*) The assumption here made, that every equation has, at least, one root, though perhaps not strictly demonstrable, has yet been sufficiently established not to admit of a doubt. — See the reflexions of Lacroix on this subject, p. 15 of the Compliment to his Elémens d'Algébre.

of x, $-(a+p)$ will necessarily be another value of it.

Which results will be found to agree with the two values of x comprised in the common formula

$$x = -\tfrac{1}{2}p \pm \sqrt{(\tfrac{1}{4}p^2 + q)}.$$

For, taking a for one of them, as for instance $-\frac{1}{2} + \sqrt{(\frac{1}{4}p^2 + q)}$, we shall have for the other

$$-a - p = +\tfrac{1}{2}p - \sqrt{(\tfrac{1}{4}p^2 + q)} - p = -\tfrac{1}{2}p - \sqrt{(\tfrac{1}{4}p^2 + q)}.$$

Which are, in effect, the two roots of the equation, as was to be shown.

It can also be farther proved, that the proposed equation

$$x^2 + px = q,$$

can have only two roots, or values of the unknown quantity.

For, if possible, let b be a third root, different from either of the former.

Then, substituting b for x, we shall have, by proceeding as above,

$$x^2 + px = b^2 + pb.$$

But, from the preceding part of the investigation, it has been shown, that

$$x^2 + px = a^2 + pa.$$

Whence, equating the right hand members of these two equations, there will arise

$$a^2 + pa = b^2 + pb, \text{ or}$$
$$a^2 - b^2 = -p(a - b).$$

And consequently, by dividing each side of this last expression by $a - b$, we shall have

$$a + b = -p, \text{ or}$$
$$b = -(a + p).$$

Which is one of the two roots before found; and therefore the supposed root b can be only a, or $-(a+p)$.

Having thus shown the truth of the proposition in this case, let us now take the general cubic equation

$$x^3+px^2+qx=r,$$

and suppose its root, or one of the values of x, to be denoted by a as before.

Then, if a be substituted for x, we shall have, by the definition,

$$a^3+pa^2+qa=r.$$

And, consequently, if this value be put for r, in the first equation, it will become

$$x^3+px^2+qx=a^3+pa^2+qa.$$

Or, by transposing the terms, and uniting those that have the same coefficients,

$$x^3-a^3+p(x^2-a^2)=-q(x-a).$$

Whence, dividing each side of this last expression by $x-a$, we shall have

$$x^2+ax+a^2+px+pa=-q, \text{ or}$$
$$x^2+(a+p)x=-a(a+p)-q.$$

Which is a quadratic equation having the two following roots,

$$x=-\tfrac{1}{2}(a+p)+\sqrt{\{\tfrac{1}{4}(a+p)^2-a(a+p)-q\}},$$
$$x=-\tfrac{1}{2}(a+p)-\sqrt{\{\tfrac{1}{4}(a+p)^2-a(a+p)-q\}}.$$

And therefore, since a, by hypothesis, is also another root, or value of x, the proposed cubic equation must necessarily have three roots.

And that it can have no more than three is

evident; because, besides the root first assumed, the quadratic equation to which it is reduced can have only two roots.

Again, let there be taken the general biquadratic equation

$$x^4 + px^3 + qx^2 + rx = s.$$

And suppose its root, or one of the values of x, as in the former instances, to be a.

Then, by substituting a for x, as before, we shall have

$$a^4 + pa^3 + qa^2 + ra = s.$$

And, consequently, if this value be put for s, in the first equation, it will become

$$x^4 + px^3 + qx^2 + rx = a^4 + pa^3 + qa^2 + ra.$$

Or, by transposing and uniting the corresponding powers of x and a, as in the last case,

$$x^4 - a^4 + p(x^3 - a^3) + q(x^2 - a^2) = -r(x - a).$$

Whence, dividing each side of this equation by $x - a$, and collecting the like terms, we shall have

$$x^3 + (a + p)x^2 + (a^2 + ap + q)x = -a(a^2 + ap + q) - r.$$

Which is a cubic equation, having, as has been before shown, three roots, or values of x.

And therefore as a, by hypothesis, is another value of x, the proposed biquadratic must necessarily have four roots, and no more, for a reason similar to that before given for cubic equations.

Hence, as it can be shown, by the same mode of reasoning, that an equation of the fifth power has five roots, and no more; one of the 6th power, six roots, and no more; and so on; it follows, that an equation of any power whatever must have as

many roots as there are units in the index of its highest term (*l*).

3. Also, since it appears from prop. 1, that every equation, when all its terms are brought to one side, is exactly divisible by the unknown quantity in that equation *minus* either of its roots, and by no other simple factor, it is evident that the equation

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \ldots + Tx + V = 0 \ldots (1)$$

of which $a, b, c, d, \ldots l$, are supposed to be its several roots, is composed of as many factors

$$(x-a)(x-b)(x-c)(x-d) \ldots (x-l) = 0 \ldots (2)$$

as the equation has roots; and that it can have no other factor whatever of that form.

Whence, as these two expressions are, by hypothesis, identical, the proposed equation, by actually multiplying the above factors, and arranging the terms according to the powers of x, will become

$$\begin{array}{l|l|l} x^m - a & x^{m-1} + ab & x^{m-2} \ldots \pm (abc \ldots l) = 0 \ldots (3) \\ \quad - b & \quad + ac & \\ \quad - c & \quad + ad & \\ \quad - d & \quad + bc & \\ \quad - \&c. & \quad + bd & \\ & \quad + cd & \\ & \quad + \&c. & \end{array}$$

(*l*) The method here given for finding the number of the roots of equations, independently of the manner in which they are usually supposed to be generated, will, it is presumed, be found far more simple and perspicuous than any that has yet been proposed: and if the mode made use of in the second part of the next proposition, for determining the relations that subsist between the roots and their coefficients could be made equally

Which form is general, whatever may be the different signs of the roots, or of the terms of the equation, taking a, b, c, A, B, C, &c. in + or − accordingly.

In which case, the following well known relations between the coefficients and roots are sufficiently obvious.

1. The sum of all the roots of any equation, having its terms arranged according to the order of the powers of the unknown quantity, is equal to the coefficient of the second term of that equation, with its sign changed.

2. The sum of the products of all the roots, taken two and two, is equal to the coefficient of the third term, with its proper sign.

3. The sum of the products of all the roots, taken three and three, is equal to the coefficient of the fourth term, with its sign changed; and so on.

4. The continued product of all the roots, is equal to the last term, taken with the same or a contrary sign, according as the degree of the equation is even or odd.

Or, if the corresponding coefficients of the two equivalent expressions (1), (3), be compared analytically, the above properties may be expressed thus:

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general, the common theory of equations would be rendered as plain and satisfactory as could be wished; but for such of them as exceed the fourth power, the process there followed, though perfectly easy and regular, becomes too extensive to be conveniently continued in the same way as for those of lower dimensions.)

$$a + b + c + d \&c. = -A$$
$$ab + ac + bc \&c. = B$$
$$abc + abd + acd \&c. = -C$$
$$\dots\dots\dots\dots$$
$$abcd \dots\dots l = \pm V$$

Where it is to be observed, that when one or more of the terms of the equation, in this or any other case, are wanting, the corresponding functions of the coefficients of those terms must be considered as $= 0$.

Hence it also follows, that any equation, wanting its second term, will have both positive and negative roots, as $a + b + c \&c.$ is then $= 0$; and that the sum of the former of these is equal to the sum of the latter.

To this we may likewise add, that, if the roots of any equation be all positive, as in formula (2); where the factors are of the form

$$(x-a)(x-b)(x-c)(x-d) \dots (x-l) = 0,$$

the signs of the terms will be alternately $+$ and $-$; since the product of an odd number of negative quantities is negative, and of an even number positive.

But if the roots be all negative, in which case the factors will be of the form

$$(x+a)(x+b)(x+c)(x+d) \dots (x+l) = 0,$$

the signs of all the terms will be positive; because the equation arises wholly from the multiplication of positive quantities.

It is also sufficiently evident, from the bare inspection of this formula, that equations of all orders, above the second, may be generated from others of an inferior order, besides simple ones.

Thus, by considering the several ways in which these factors can be combined, it is plain, that a cubic equation may be formed from the multiplication of a quadratic and a simple equation; a biquadratic from the multiplication of two quadratics, or of a cubic and a simple equation; and so on.

Or, the relations which have been thus found to subsist between the coefficients and roots of any equation, may be derived from the second proposition here given, independently of the first, as follows:

Let a, b, be the two roots of the general quadratic equation,

$$x^2+px+q=0;$$

Then, by the definition given at the head of the present article, we shall have

$$a^2+pa+q=0 \quad . \; . \; . \quad (1)$$
$$b^2+pb+q=0 \quad . \; . \; . \quad (2)$$

And, consequently, by subtracting the latter of these two equations from the former, there will arise

$$a^2-b^2+p(a-b)=0.$$

Whence, dividing this last equation by $a-b$, we shall have

$$a+b+p=0, \text{ or } p=-(a+b)$$

And, consequently, if this value of p be substituted in either of the equations (1), (2), it will give

$$a^2-a(a+b)+q=0; \text{ or } q=ab.$$

Which results agree with those before obtained for the sum and product of the two roots of the proposed equation.

Again, let a, b, c, be the three roots of the general cubic equation

$$x^3+px^2+qx+r=0.$$

Then, according to the definition before mentioned, we shall have

$$a^3+pa^2+qa+r=0 \quad \ldots (1)$$
$$b^3+pb^2+qb+r=0 \quad \ldots (2)$$
$$c^3+pc^2+qc+r=0 \quad \ldots (3)$$

And, consequently, by taking the difference between the first of these equations and each of the other two, there will arise

$$a^3-b^3+p(a^2-b^2)+q(a-b)=0,$$
$$a^3-c^3+p(a^2-c^2)+q(a-c)=0.$$

Whence, if the first of these two last equations be divided by $a-b$, and the second by $a-c$, we shall have

$$a^2+ab+b^2+p(a+b)+q=0 \quad \ldots (4)$$
$$a^2+ac+c^2+p(a+c)+q=0 \quad \ldots (5)$$

And, therefore, by subtracting the last of these two equations from the first, the result will be

$$a(b-c)+(b^2-c^2)+p(b-c)=0.$$

Which, being divided by the common factor $b-c$, gives

$$a+b+c+p=0; \text{ or } p=-(a+b+c).$$

Wherefore, by substituting this value of p, in either of the equations (4), (5), there will arise

$$q=ab+ac+bc.$$

And if the values of p and q, thus found, be substituted in either of the equations (1), (2), (3), we shall have

$$r=-abc.$$

Which results also agree with those before found for cubics.

Lastly, let a, b, c, d, be the four roots of the general biquadratic equation

$$x^4+px^3+qx^2+rx+s=0.$$

Then, as in the former instances, we shall have

$$a^4+pa^3+qa^2+ra+s=0 \quad . \; . \; . \quad (1)$$
$$b^4+pb^3+qb^2+rb+s=0 \quad . \; . \; . \quad (2)$$
$$c^4+pc^3+qc^2+rc+s=0 \quad . \; . \; . \quad (3)$$
$$d^4+pd^3+qd^2+rd+s=0 \quad . \; . \; . \quad (4)$$

And if the second, third and fourth of these equations be subtracted from the first, there will arise

$$a^4-b^4+p(a^3-b^3)+q(a^2-b^2)+r(a-b)=0,$$
$$a^4-c^4+p(a^3-c^3)+q(a^2-c^2)+r(a-c)=0,$$
$$a^4-d^4+p(a^3-d^3)+q(a^2-d^2)+r(a-d)=0.$$

Whence, if the first of these three last equations be divided by $a-b$, the second by $a-c$, and the third by $a-d$, we shall have

$$a^3+a^2b+ab^2+b^3+p(a^2+ab+b^2)+q(a+b)+ r=0 \quad . \; . \; . \; . \; . \; . \quad (5)$$
$$a^3+a^2c+ac^2+c^3+p(a^2+ac+c^2)+q(a+c)+ r=0 \quad . \; . \; . \; . \; . \; . \quad (6)$$
$$a^3+a^2d+ad^2+d^3+p(a^2+ad+d^2)+q(a+d)+ r=0 \quad . \; . \; . \; . \; . \; . \quad (7)$$

And consequently, by subtracting the second and third of these last equations from the first, there will arise

$$a^2(b-c)+a(b^2-c^2)+(b^3-c^3)+p\{a(b-c)+b^2-c^2\} + q(b-c)=0,$$
$$a^2(b-d)+a(b^2-d^2)+(b^3-d^3)+p\{a(b-d)+b^2-d^2\} + q(b-d)=0.$$

Wherefore, dividing the first of these by $b-c$, and the second by $b-d$, we shall have

$$a^2+a(b+c)+b^2+bc+c^2+p(a+b+c)+ q=0 \quad . \; . \; . \; . \; . \; . \quad (8)$$
$$a^2+a(b+d)+b^2+bd+d^2+p(a+b+d)+ q=0 \quad . \; . \; . \; . \; . \; . \quad (9)$$

And, lastly, by taking the second of these two equations from the first, there will remain

$$a(c-d)+b(c-d)+(c^2-d^2)+p(c-d)=0.$$

Whence, dividing this result by $c-d$, we shall have

$$a+b+c+d+p=0;\ \text{or}\ p=-(a+b+c+d).$$

And, by substituting this value of p in either of the equations (8), (9), the result will give

$$q=ab+ac+ad+bd+bc+dc.$$

Also, if the values of p and q, thus obtained, be substituted in either of the equations (5), (6), (7), there will arise

$$r=-(abd+abc+adc+bdc).$$

And these values of p, q, r, being again substituted in either of the equations (1), (2), (3), (4), will give

$$s=abcd.$$

Which results are likewise the same as those before found for biquadratics.

And by proceeding in the same manner with equations of the 5th, 6th, and other higher dimensions, the relations, before mentioned, between their roots and coefficients, will be still found to subsist (*m*).

Hence if there be given the following cubic equation,

$$x^3+2x^2-23x-60=0,$$

(*m*) The relations above established between the roots of any equation and its coefficients, will be found insufficient for the determination of the roots themselves; since the several equations, formed according to these conditions, will always give, by elimination, results similar to those marked (1), (2), (3), (4); which, being of the same form as the proposed equation, must evidently be attended with equal difficulties in their solution.

of which the roots are readily found by trial to be

$$+5, \quad -4, \text{ and } -3,$$

we shall have, according to the properties above mentioned,

$$5-4-3=-2,$$
$$-20-15+12=-23,$$
$$5\times -4\times -3=+60.$$

And, if there be taken the incomplete cubic equation

$$x^3-19x+30=0,$$

of which the three roots, found as above, are

$$+2, \quad +3, \text{ and } -5.$$

we shall likewise have, according to the same properties,

$$2+3-5=0,$$
$$6-10-15=-19,$$
$$2\times 3\times -5=-30.$$

4. By considering the subject in this way, we can also readily find the sum of the squares, the sum of the cubes, and other symmetrical functions of the roots of any equation, in terms of the coefficients of that equation, without knowing the roots themselves.

Thus, let there be taken the general equation

$$x^m+\mathrm{A}x^{m-1}+\mathrm{B}x^{m-2}+\mathrm{C}x^{m-3}+\ldots+\mathrm{T}x+\mathrm{V}=0,$$

of which the several roots are supposed to be $a, b, c, d, \ldots l$. And for the sake of abbreviating the results, let

$$a+b+c+d \ldots\ldots +l=\mathrm{S}_1$$
$$a^2+b^2+c^2+d^2 \ldots\ldots +l^2=\mathrm{S}_2$$
$$a^3+b^3+c^3+d^3 \ldots\ldots +l^3=\mathrm{S}_3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a^{m-1}+b^{m-1}+c^{m-1}+\ldots +l^{m-1}=\mathrm{S}_{m-1}$$
$$a^m+b^m+c^m+d^m \ldots +l^m=\mathrm{S}_m$$

Then, if a, b, c, d, . . . l, be successively substituted for x, we shall have, according to the definition before mentioned,

$$a^m + \text{A}a^{m-1} + \text{B}a^{m-2} + \ldots + \text{T}a + \text{V} = 0$$
$$b^m + \text{A}b^{m-1} + \text{B}b^{m-2} + \ldots + \text{T}b + \text{V} = 0$$
$$c^m + \text{A}c^{m-1} + \text{B}c^{m-2} + \ldots + \text{T}c + \text{V} = 0$$
$$d^m + \text{A}d^{m-1} + \text{B}d^{m-2} + \ldots + \text{T}d + \text{V} = 0$$
$$\ldots\ldots\ldots\ldots$$
$$l^m + \text{A}l^{m-1} + \text{B}l^{m-2} + \ldots + \text{T}l + \text{V} = 0$$

And, consequently, by adding these results together, there will arise the following general expression,

$$\text{S}_m + \text{A}\text{S}_{m-1} + \text{B}\text{S}_{m-2} + \text{C}\text{S}_{m-3} + \ldots + \text{T}\text{S}_1 + m\text{V} = 0;$$

which involves only the coefficients of the given equation, and the sums of the several powers of its roots.

Whence, as this formula comprehends every case of these powers not exceeding those of the mth degree, if m be taken $= 1, 2, 3$, &c. respectively, mV will necessarily become $= \text{A}, 2\text{B}, 3\text{C}$, &c. and we shall have

$$\text{S}_1 + \text{A} = 0$$
$$\text{S}_2 + \text{A}\text{S}_1 + 2\text{B} = 0$$
$$\text{S}_3 + \text{A}\text{S}_2 + \text{B}\text{S}_1 + 3\text{C} = 0$$
$$\text{S}_4 + \text{A}\text{S}_3 + \text{B}\text{S}_2 + \text{C}\text{S}_1 + 4\text{D} = 0 \ (n)$$
&c. &c.

Where the law by which the next succeeding equations may be formed is sufficiently obvious.

(n) These elegant formulæ, which were first given by Newton, without demonstration, in his Universal Arithmetic, have since been investigated in various ways, both by our own mathematicians and those of the continent; but it is conceived, that the process from which they are here derived, possesses a degree of simplicity and conciseness superior to any that has yet been devised.

And if the values of s_1, s_2, s_3, &c. in the first, second, third, &c. of these expressions, be substituted in each of the following ones, the above formulæ may be exhibited as follows:

$$s_1 = a + b + c + \ldots + l = -A \quad \ldots \quad (1)$$
$$s_2 = a^2 + b^2 + c^2 + \ldots + l^2 = A^2 - 2B$$
$$s_3 = a^3 + b^3 + c^3 + \ldots + l^3 = -A^3 + 3AB - 3C$$
$$s_4 = a^4 + b^4 + c^4 + \ldots + l^4 = A^4 - 4A^2B + 4AC + 2B^2 - 4D$$
&c. &c.

From which it appears, that the sums of the several entire and positive powers of the roots of any equation, may be expressed in rational functions of the coefficients of that equation.

Also, by reversing the former set of these expressions, we shall have the following forms:

$$A = -s_1$$
$$B = -\frac{As_1 + s_2}{2}$$
$$C = -\frac{Bs_1 + As_2 + s_3}{3}$$
$$D = -\frac{Cs_1 + Bs_2 + As_3 + s_4}{4}$$
&c. &c.

By means of which we can readily find the coefficients of any equation, when the sums of the several powers of its roots are known.

And if it should be required to find the sums of the powers of the roots that are greater than the index m, a similar theorem, for this purpose, may be easily derived from the expression first given.

For if the proposed equation be multiplied by x^n, it will then become of the form

$$x^{m+n} + Ax^{m+n-1} + Bx^{m+n-2} + \ldots + Tx^{n+1} + Vx^n = 0.$$

Where, the roots being the same as before, we shall have, by successively substituting a, b, c, d, &c. in the place of x, and adding the results together, as in the former case,

$$s_{m+n} + As_{m+n-1} + Bs_{m+n-2} + \ldots\ldots Ts_{m+1} + Vs_m = 0.$$

Whence, if n, in this last formula, be taken $=0$, 1, 2, 3, &c. respectively, the values of s_m, s_{m+1}, s_{m+2}, &c. may be readily obtained, as before.

A similar formula may, also, be easily derived, in the same way, for finding the sums of the reciprocals of the several powers of the roots of any equation.

For, if $\frac{1}{y}$ be substituted for x, in the proposed equation, we shall have, after a proper reduction of the terms,

$$y^m + \frac{T}{V}y^{m-1} + \frac{S}{V}y^{m-2} + \frac{R}{V}y^{m-3} + \ldots\ldots + \frac{A}{V}y + \frac{1}{V} = 0.$$

And therefore, since $y = \frac{1}{x}$, or x^{-1}, this formula will be the same for the negative powers of the roots, as the former is for their positive powers.

Whence, denoting these powers by s_{-1}, s_{-2}, s_{-3}, &c. and following a process similar to that before used, there will arise, in this case,

$$s_{-1} + \frac{T}{V} = 0$$

$$s_{-2} + \frac{T}{V}s_{-1} + \frac{2}{V}S = 0$$

$$s_{-3} + \frac{T}{V}s_{-2} + \frac{S}{V}s_{-1} + \frac{3}{V}R = 0$$

$$s_{-4} + \frac{T}{V}s_{-3} + \frac{S}{V}s_{-2} + \frac{R}{V}s_{-1} + \frac{4}{V}Q = 0$$

&c. &c.

Or, substituting these values of s_{-1}, s_{-2}, &c. in each of the succeeding forms, as in the instance before given, we shall have

$$s_{-1} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \&c. = -\frac{T}{V} \quad . \quad . \quad . \quad . \quad (2)$$

$$s_{-2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \&c. = \frac{T^2 - 2VS}{V^2}$$

$$s_{-3} = \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \&c. = -\frac{T^3 - 3VST + 3RV^2}{V^3}$$

$$s_{-4} = \frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} \&c. = \frac{T^4 - 3VST^2 + 4TRV^2 + 2S^2V^2 - 4QV^3}{V^4} (o)$$

&c. &c.

Where it may be observed, that all the entire negative powers of the roots of any equation, as well as those that are positive, can be expressed in rational functions of the coefficients of that equation.

As an example of the application of this method, let there be taken the following equation of the fourth degree, which is that given by Newton for the same purpose,

$$x^4 - x^3 - 19x^2 + 49x - 30 = 0.$$

Then, since $A = -1$, $B = -19$, $C = +49$, and $D = -30$, we shall have by formulæ (1)

(o) By means of the above formulæ we might also have shown, that any rational and symmetrical algebraical function whatever of the roots of any equation, as $a^n b^p c^q$ &c. + &c. can be expressed in terms of the coefficients of that equation; and that they may be advantageously employed in determining the nearest approximate roots of equations, as well as in various other analytical investigations. But these, and other similar researches, would be incompatible with the limited plan of the present performance, which is meant to comprehend such of the most useful and interesting parts of the science only, as can be brought into a moderate compass.

$$s_1 = 1,\ s_2 = 39,\ s_3 = -89,\ s_4 = 723.$$

To which we may farther add, by calculating the succeeding steps,

$$s_5 = -2849,\ s_6 = -16419,\ \&c.$$

And taking the values of D, C, B, A, for those of V, T, S, R, in the formulæ (2) last given, we may readily obtain, in a similar way, the sums of the powers of the negative roots of this equation.

5. Any equation, having fractional coefficients, may be transformed into another, that shall have the coefficient of its first term unity, and those of the rest, as well as the absolute term, whole numbers (*p*).

For taking, instead of a general equation of this kind, the following partial example,

$$x^3 + \frac{1}{2}x^2 + \frac{2}{3}x + \frac{3}{4} = 0,$$

which will be sufficient to show the method that should be followed in other cases.

It is obvious, that if each of the terms be multiplied by the least common multiple of all the denominators, we shall have

$$12x^3 + 6x^2 + 8x + 9 = 0,$$

where the coefficients and absolute term are all whole numbers.

(*p*) The last term of any equation, agreeably to what is here said, is frequently called the absolute term of that equation; but this is unnecessary, since it may always be considered as making a part of the coefficients as well as the rest; for x^0 being equal to unity, the last term is evidently the coefficient of x^0.

And if $12x$, in this case, be put $=y$, or $x=\frac{y}{12}$, there will arise

$$\frac{y^3}{12^3}+6\left(\frac{y^2}{12^2}\right)+8\left(\frac{y}{12}\right)+9=0.$$

Which last equation, when all the terms are multiplied by 12^2, gives

$$y^3+6y^2+96y+1296=0,$$

where the coefficient of the first term is unity, and those of the rest whole numbers, as was required.

So that when the value of y in this last equation is known, we shall have, for the proposed equation,

$$x=\frac{y}{12}.$$

6. No equation, of which the coefficient of the first term is unity, and the rest of the coefficients whole numbers, can have a fractional root.

For, taking, as before, any general equation,

$$x^m+\mathrm{A}x^{m-1}+\mathrm{B}x^{m-2}+\ldots\ldots+\mathrm{T}x+\mathrm{V}=0.$$

Let, if possible, the root, or value of x, be the irreducible fraction $\frac{a}{b}$.

Then, by substituting this value for x, in the given equation, there will arise

$$\frac{a^m}{b^m}+\mathrm{A}\frac{a^{m-1}}{b^{m-1}}+\mathrm{B}\frac{a^{m-2}}{b^{m-2}}+\ldots\ldots+\mathrm{T}\frac{a}{b}+\mathrm{V}=0.$$

And therefore, by multiplying each of the terms by b^{m-1}, and then carrying all the results, except the first, to the right hand side, we shall have

$$\frac{a^m}{b}=-\{\mathrm{A}a^{m-1}+\mathrm{B}ba^{m-2}+\mathrm{C}b^2a^{m-3}+\ldots\ldots$$
$$+\mathrm{T}b^{m-2}a+\mathrm{V}b^{m-1}\}.$$

Where, since m, a, b, A, B, &c. are all whole numbers, the entire right hand member of this identity must also be a whole number.

But, by hypothesis, $\frac{a}{b}$, and consequently $\frac{a^m}{b}$, (1), Art. 2, is an irreducible fraction; whence, if $\frac{a}{b}$ were a root of the proposed equation, we should have a fraction, in its lowest terms, equal to a whole number; which is absurd.

7. Any equation may be transformed into another, that shall have its roots of the same value as those of the former, but with contrary signs.

For supposing, in the first place, the given equation to be of an odd number of dimensions, as

$$x^{2m+1} + Ax^{2m} + Bx^{2m-1} + \ldots\ldots + Tx + V = 0. \ldots (1)$$

Then, substituting $-x$ in the place of $+x$, the equation will become

$$-x^{2m+1} + Ax^{2m} - Bx^{2m-1} + \ldots\ldots - Tx + V = 0.$$

And, consequently, by taking all the terms to the other side of the equation; or, which is the same thing, by changing their signs, we shall have

$$x^{2m+1} - Ax^{2m} + Bx^{2m-1} - \ldots\ldots + Tx - V = 0. \ldots (2)$$

Where it is plain, from what has been shown in Art. 3, that the positive roots of equation (1) will become negative in equation (2), and *vice versa*.

And if the proposed equation be of an even number of dimensions, as

$$x^{2m} + Ax^{2m-1} + Bx^{2m-2} + \ldots\ldots + Tx + V = 0, \ldots (3)$$

we shall have, by substituting $-x$ in the place of $+x$, as before,

$$x^{2m} - Ax^{2m-1} + Bx^{2m-2} - \ldots\ldots - Tx + V = 0. \ldots (4)$$

In which case, also, it appears, that the roots of this equation will have opposite signs to those of equation (3), and the contrary.

Hence, from this investigation, it follows, that the positive roots of any equation may be changed into negative roots; and the negative roots into positive ones, of the same value, by barely changing the signs of all the alternate terms, beginning with the second.

Thus the equation $x^3-2x^2-5x+6=0$, having the negative root -2, and the positive roots 1 and 3, will be changed, by this means, into the equation

$$x^3+2x^2-5x+6=0,$$

which has the positive root $+2$, and the negative roots -1 and -3.

Also, the incomplete equation $x^3-19x-30=0$, having the negative roots -2, -3, and the positive root 5, may be changed, when put under the form

$$x^3+0-19x-30=0, \text{ to}$$
$$x^3-19x+30=0;$$

which last equation has the negative root -5, and the positive roots 2 and 3.

8. Any equation may be transformed into another, the roots of which shall be greater or less than those of the former by a given quantity.

Thus, let there be taken, as before, the following general equation,

$$x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3}+\ \ldots\ldots\ +Tx+V=0.$$

And suppose it were required to transform it into another, whose roots shall be greater than those of the given equation by h.

Then, if y be made to represent one of these roots, we shall have, by the nature of the question,

$$y = x + h, \text{ or } x = y - h.$$

And consequently, by substituting $y - h$ for x, in the proposed equation, there will arise . . . (1)

$$\begin{array}{l|l|l|l} y^m - mh & y^{m-1} + \frac{m(m-1)}{2}h^2 & y^{m-2} - \frac{m(m-1)(m-2)}{2 \cdot 3}h^3 & y^{m-3} \text{\&c.} = 0 \\ + A & - (m-1)Ah & + \frac{(m-1)(m-2)}{2}Ah^2 & \\ & + B & - \frac{m-2}{1}Bh & \\ & & + C & \end{array}$$

Which equation will evidently fulfil the conditions required, y being greater than x by h.

And if y be taken $= x - h$, or $x = y + h$, we shall readily obtain, by a similar substitution, an equation whose roots are less than those of the given equation by h.

Whence, also, as h, in the above case, is indeterminate, this mode of substitution may be used for destroying one of the terms of the proposed equation.

For, putting the coefficient $-mh + A = 0$, we shall have

$$h = \frac{A}{m}, \text{ and } x = y - \frac{A}{m}.$$

From which it follows, that the second term of any equation may be taken away, by substituting for the unknown quantity some other unknown quantity, augmented by the coefficient of the second term, taken with a contrary sign, and divided by the index of the highest power of the equation.

The third term of the proposed equation may

also be taken away by means of the coefficient, or formula

$$\frac{m(m-1)}{2}h^2-(m-1)\text{A}h+\text{B}=0.$$

Where the determination of h requires the solution of an equation of the second degree.

And, from the same general expression, it appears, that the value of h, necessary for destroying the fourth term, can be determined by an equation of the third degree; and so on.

Whence it follows, that the extermination of the last term of any equation depends upon the solution of an equation of the same degree with the proposed equation; and must, therefore, be attended with the same difficulties (q).

In like manner, also, any defective equation, when necessary, may be rendered complete in all its terms, by substituting $y \pm h$ for x, as above.

9. Every equation may be transformed into another, the roots of which shall be any multiples or parts of those of the former.

Thus, let there be taken, as in the former instances, the general equation

$$x^m+\text{A}x^{m-1}+\text{B}x^{m-2}+\text{C}x^{m-3}+\ \ldots\ +\text{T}x+\text{V}=0,$$

And, in order to convert it into another, whose

(q) It may be imagined, that by taking $x=y+h+k$, instead of $y+h$, in the above process, it would be possible to destroy two terms of the proposed equation at once; but it must be observed, that by substituting $h+k$ for h in equation (1), and then putting the coefficients of any two of its terms $=0$, we shall have two equations between the same sum $h+k$; which, except in some particular cases, will not enable us to determine either h or k.

roots shall be some multiple of those of the given equation, let there be put

$$y = ex, \text{ or } x = \frac{y}{e}.$$

Then, by substituting this value for x in the proposed equation, there will arise

$$\frac{y^m}{e^m} + A\frac{y^{m-1}}{e^{m-1}} + B\frac{y^{m-2}}{e^{m-2}} + \ldots + T\frac{y}{e} + V = 0.$$

And, consequently, if this be multiplied by e^m, we shall have

$$y^m + Aey^{m-1} + Be^2y^{m-2} + \ldots Te^{m-1}y + Ve^m = 0 \ldots (1)$$

Which equation will evidently fulfil the conditions required, y being equal to ex.

And if y be put $= \frac{x}{e}$, or $x = ey$, we shall readily obtain, by a similar substitution of this value for x, the equation

$$y^m + \frac{A}{e}y^{m-1} + \frac{B}{e^2}y^{m-2} + \ldots + \frac{T}{e^{m-1}}y + \frac{V}{e^m} = 0,$$

where the roots are equal to those of the proposed equation divided by e.

Or, if this value of x be substituted in the original equation, as above, the roots of the resulting equation may be made equal to any part of those of the former.

A similar transformation may also be used for rendering either of the coefficients of any equation equal to unity, or to any other given number; which may evidently be done by putting

$$Ae, \ Be^2, \text{ or } Ce^3, \ \&c. = 1, \text{ or } n,$$

and then finding the value of e in terms of 1 or n, and the known coefficients A, B, C, &c.

Thus, if it were required to transform the equation

$$x^3 - ax = b$$

into another that shall have 1 for the coefficient of the unknown quantity, instead of a,

We shall have, by putting $x = zy$, $z^3y^3 - azy = b$; or, dividing by z^3,

$$y^3 - \frac{a}{z^2}y = \frac{b}{z^3}.$$

And, consequently, if z be now taken $= \sqrt{a}$, this equation will become

$$y^3 - y = \frac{b}{a\sqrt{a}}.$$

Or, putting, for the sake of simplifying the expression, $\frac{b}{a\sqrt{a}} = r$,

$$y^3 - y = r.$$

Where it may be observed, that a cubic equation, when brought to this form, can only fall under the irreducible case, when $\frac{1}{27} = \frac{1}{4}r^2$, or

$$r = \frac{2}{3}\sqrt{\frac{1}{3}}, \text{ or } \frac{2}{9}\sqrt{3},$$

which can only happen when the positive value of y lies between the limits 1 and $\frac{2}{3}\sqrt{3}$ (r).

10. In nearly the same manner, also, may any equation be transformed into another, the roots of

(r) This principle is used by Clairaut, in his *Elémens d'Algébre*, for obtaining a more rapid approximation towards the root of equations falling under the irreducible case of cubics, than by the common method; and may be turned to a good account, in the formation of tables for the ready solution of all questions of this kind.

which shall be the reciprocals of the roots of the former.

For taking the same equation as above, and putting

$$x=\frac{1}{y},$$

the transformed equation, when its several terms are multiplied by the highest power of y, will give

$$\text{V}y^m+\text{T}y^{m-1}+\text{S}y^{m-2}+\ \ldots\ +\text{A}+1=0.$$

And as the order of the coefficients, in this case, is reversed, if the second term of the given equation be wanting, the last but one of the transformed equation will be wanting; and so on.

It may here, also, be observed, that the greatest root y, of the transformed equation, will correspond with the least root of the given equation; and reciprocally.

11. It can, likewise, be farther shown, that any equation may be converted into another, the roots of which shall be equal to the square roots of those of the former.

For taking the required relation, as expressed by

$$y=\sqrt{x},\ \text{or}\ x=y^2,$$

we shall have, by substituting this value in the proposed equation,

$$y^{2m}+\text{A}y^{2m-2}+\text{B}y^{2m-4}+\ \ldots\ \text{T}y^2+\text{V}=0.$$

In which expression, all the terms involving the odd powers of x are wanting, owing to $\sqrt{x}$ being affected with the double sign $+$ and $-$; and consequently $y=\sqrt{x}$.

And, in nearly the same way, may any equation be converted into another, the roots of which shall be the squares of those of the former.

12. The superior limit of the positive roots of any equation is the greatest negative coefficient of that equation, taken positively, and augmented by unity; and the inferior limit is the fraction arising from dividing the last term of the equation, by the same term, augmented by the greatest coefficient that has a contrary sign to that term, taking them all positively (s).

For let the equation of which the limits of the roots are to be determined be

$$x^m - Ax^{m-1} + Bx^{m-2} - Cx^{m-3} + \ldots - Tx + V = 0 \ldots (1),$$

Then, if there can be found a value of x, which will render the sum of all the positive terms of the polynomial

$$x^m - Ax^{m-1} + Bx^{m-2} - Cx^{m-3} + \ldots - Tx + V,$$

greater than the sum of all the negative terms, it is plain that this value must be greater than the greatest positive root of the given equation; which root would only give a result $=0$.

Take, therefore, as the most unfavourable case for this purpose, that in which all the coefficients A, B, C, &c. are negative, and each equal to the greatest, which suppose to be N.

Then, if by any assumption for the value of x, the polynomial

(s) By the *superior limit* of the positive roots of an equation, is to be understood a number that exceeds the greatest positive root of that equation; and by the *inferior limit*, is meant a number that is less than the least positive root. Thus, the roots of the equation $x^2 - 7x + 12 = 0$, being $+3$ and $+4$, any number greater than 4 is considered as the superior limit, and any number less than 3 as the inferior limit.

$$x^m - Nx^{m-1} - Nx^{m-2} - Nx^{m-3} - \ldots - Nx - N$$

can be made positive, the former expression must also be positive; and will consequently fulfil the conditions of the question.

But this last polynomial, by collecting all its terms together, except the first, is evidently equivalent to

$$x^m - N(x^{m-1} + x^{m-2} + x^{m-3} + \ldots + x + 1);$$

Or, by the nature of geometrical progression, pointed out in ART. (M), to the simple expression,

$$x^m - N\left(\frac{x^m - 1}{x - 1}\right), \text{ or } \frac{x^m(x - 1 - N) + N}{x - 1}$$

Whence, if x be here taken equal to $N + 1$, the result will become $\frac{N}{N}$, or $+1$, which being positive, the superior limit required will evidently be $N + 1$ (*t*).

In like manner, we may also readily find the inferior limit of the least positive root of the given equation, by means of the relation

$$x = \frac{1}{y}.$$

For, if this value be substituted for x in equation (1), we shall have an equation in y, in which, when reduced to the same form as the original equation, the greatest negative coefficient, taken positively, and augmented by 1, will be the superior limit of the positive values of y.

(*t*) By a similar process it may be shown, that the value of the unknown quantity x in any polynomial

$$x^m + ax^{m-1} + bx^{m-2} + cx^{m-3} + \ldots tx + v$$

that will render its first term greater than the sum of all the following terms, is the greatest of all the coefficients of that polynomial (abstracting from their signs) augmented by 1.

And therefore, since it appears, from the above relation, that x in the proposed equation diminishes as y in the transformed equation increases, it is plain, that the superior limit of y, thus obtained, will be the inferior limit of x.

Whence, supposing N' to be the greatest negative coefficient of the transformed equation in y, the inferior limit of the positive values of x in the given equation will be

$$\frac{1}{N'+1}.$$

Or, if it should be required to determine the inferior limit of the positive roots immediately, in terms of the coefficients of the proposed equation,

$$x^m - Ax^{m-1} + Bx^{m-2} + \ldots + Px^{m-n} \ldots - Nx^{m-r} \ldots \pm V = 0 \ldots (3)$$

this may be readily done by means of the transformed equation

$$y^m \ldots - \frac{N}{\pm V} y^r \ldots + \frac{P}{\pm V} y^n \ldots + 1 = 0 \ldots (4)$$

For, supposing P and N to be the greatest positive and negative coefficients of the proposed equation, the greatest negative coefficient of the transformed equation will be

$$\frac{N}{V} \text{ for } +V, \text{ and } \frac{P}{V} \text{ for } -V.$$

Hence, substituting the former of these values for N', in the expression above given, the inferior limit, in the first of these cases, will be

$$\frac{1}{N'+1} = \frac{V}{V+N}.$$

And, by a like substitution of the latter value for N', the limit in the second case will be

$$\frac{1}{N'+1}=\frac{V}{V+P}.$$

Which results agree with those that are stated in the proposition.

Thus, if there be taken, as a practical example of this rule, the cubic equation

$$x^3-5x^2+7x+2=0,$$

the superior limit of all its positive roots will be $5+1$, and their inferior limit $\frac{2}{2+5}$.

The rules above given are universally applicable, in whatever part of the equation the greatest negative coefficient may occur; but if two or more of its first terms be positive, other limits of the roots can be found that will approach nearer to their true values than the former (*u*).

For let there be taken any general equation of the form

$$x^m+Ax^{m-1}+\ldots+Qx^{m-n+1}-Rx^{m-n}+\ldots+Tx\pm\ldots+V=0 \quad \ldots (2)$$

in which the first term, that is affected with the negative sign, is $-Rx^{m-n}$.

Then, suppressing all the intermediate terms between x^m and Rx^{m-n}, and taking each of the rest with the greatest negative coefficient N, we shall only have to render the polynomial

(*u*) It may here be observed, that in any cubic equation of the form $x^3-ax+b=0$, the $\sqrt{a}$ will be greater than the greatest root. For if this be substituted for x, we shall have $a\sqrt{a}-a\sqrt{a}+b=b$; which gives a positive result instead of 0.

And if the proposed cubic be of the form $x^3-ax-b=0$, it will be found, by a similar substitution, that $\sqrt{a}$ will be less than its least root.

$$x^m - \text{N}(x^{m-n} + x^{m-n-1} + x^{m-n-2} + \ldots + x + 1)$$

a positive quantity, as in the instance above given.

But this, by the nature of geometrical progression, is equivalent to

$$x^m - \text{N}\left(\frac{x^{m-n+1}-1}{x-1}\right), \text{ or } x^m - \text{N}\left(\frac{x^{m-n+1}}{x-1}\right) + \frac{\text{N}}{x-1},$$

which last expression will evidently be positive, when

$$x^m \asymp \text{N}\left(\frac{x^{m-n+1}}{x-1}\right), \text{ or } (x-1)x^{n-1} \asymp \text{N}.$$

Let, therefore, $x - 1 = z$; and we shall now have to make $z(z+1)^{n-1} \asymp \text{N}$; which it will manifestly be when

$$z(z^{n-1}), \text{ or } z^n = \text{N}; \text{ or } z = \sqrt[n]{\text{N}}.$$

Hence, since $x = 1 + z$, the superior limit of the positive roots of the proposed equation, in this case, will be

$$1 + \sqrt[n]{\text{N}};$$

where n is equal to the number of terms standing before that which is affected by the first negative sign.

And if $\frac{1}{y}$ be substituted for x in equation (2), the inferior limit, found as in the first case, will be

$$\frac{1}{1 + \sqrt[n]{\text{N}'}};$$

where N′ denotes the greatest negative coefficient of the transformed equation in y, and n the number of terms preceding the first negative term, as before.

Or, taking P and N for the greatest positive and negative roots of the given equation, the same limit, when expressed directly, in these terms, will be

$$\frac{V}{V+\sqrt[n]{N}}, \text{ or } \frac{V}{V+\sqrt[n]{P}},$$

according as the last term V is affirmative or negative.

Thus, if we take, as a practical example of this method, the equation

$$x^6+2x^5+3x^4+x^3-7x^2-81x+30=0,$$

the superior limit of its positive roots will be $1+\sqrt[4]{81}$, or 4; whereas, by the first rule, it would have been 82.

And if there be taken the following equation,

$$x^5+2x^4-50x^3-100x^2+49x+98=0,$$

the superior limit will be $1+\sqrt{100}$, or 11, and the inferior limit $\frac{98}{98+\sqrt{100}}$, or $\frac{49}{54}$; whereas, by the former rule, they would be 101, and $\frac{98}{98+100}$, or $\frac{49}{99}$.

In like manner, if it were required to determine the limits of the negative roots of the equation

$$x^m-Ax^{m-1}+Bx^{m-2}-Cx^{m-3}+\ldots-Tx+V=0.$$

This may be done by barely substituting $-x$ for x, and then finding the limits of the positive roots of this last equation, which will evidently be those of the negative roots of the former.

For this substitution produces no other alteration, either in the roots or the coefficients of the given equation, than that of changing their signs (Art. 7).

Hence, it is farther evident, that the superior limit of all the roots of any equation, abstracting from their signs, is the greatest coefficient of that equation augmented by 1; and their inferior limit is the fraction arising from dividing the last term

of the equation by the same term, augmented by the coefficient above mentioned.

Thus, the superior limit of all the roots of the equation

$$x^4 + 3x^3 + 5x^2 - 4x - 2 = 0,$$

abstracting from their signs, is $5 + 1$, or 6, and their inferior limit is $\frac{2}{1+5}$, or $\frac{1}{3}$; though it is to be observed, that the limits of the roots, obtained in this way, often differ considerably from their true values.

Newton, in his Universal Arithmetic, (p. 204, English edition,) has given a method of determining the limits of the roots above mentioned, which, in many cases, will bring them much nearer to the truth than either of those here proposed; but it has not the advantage which these possess, of being free from trials, and is, besides, too tedious to be generally employed for equations of the higher orders.

The method consists in transforming the given equation into another (as in Art. 8) that shall have its roots less than those of the former by some quantity e, and then taking e in the resulting equation so, that all the coefficients of that equation shall be positive; in which case the assumed number e will necessarily be greater than the greatest root of the proposed equation.

Thus, as a partial example of this method, let it be required to find a number greater than the greatest root of the equation

$$x^3 - 4x^2 + 6x - 1 = 0.$$

Here, by assuming $x=y+e$, we shall have for the limiting equation

$$\left.\begin{array}{r} y^3+3ey^2+3e^2y+e^3 \\ -4y^2-8ey-4e^2 \\ +6y+6e \\ -1 \end{array}\right| =0$$

Where, if 2 be substituted for e, each of the quantities e^3-4e^2+6e-1, $3e^2-8e+6$, and $3e-4$, will be positive; and consequently 2 will be greater than the greatest value of x.

In like manner, if the signs of the second, fourth, &c. terms be changed, a limit greater than the greatest root of the resulting equation will be less than the least root of the proposed equation.

Thus, if it were required to find a limit less than the least root of the equation

$$x^3 \pm 0-7x+35, \text{ or}$$
$$x^3-7x+35=0.$$

This, by changing the signs of the alternate terms, becomes

$$x^3-7x-35=0.$$

Whence, assuming $x=y+e$, the limiting equation will be

$$\left.\begin{array}{r} y^3+3ey^2+3e^2y+e^3 \\ -7y-7e \\ -35 \end{array}\right| =0$$

Where, if 4 be substituted for e, every term will become positive; and consequently 4 is greater than the greatest root of the equation

$$x^3-7x-35=0,$$

and -4 less than the least root of the equation

$$x^3 - 7x + 35 = 0 \text{ (x)}.$$

13. If two different numbers, when substituted for the unknown quantity, in any equation, give results with contrary signs, one of the real roots of the equation will always be comprised between these numbers.

For let there be taken, as in the cases before given, the general equation

$$x^m - Ax^{m-1} + Bx^{m-2} - Cx^{m-3} + \ldots - Tx + V = 0.$$

Which, by putting P equal to the sum of all the positive terms, and N equal to the sum of the negative terms, will become of the form

$$P - N = 0.$$

Then, if some positive number p, when substituted for x, gives a negative result, and some greater nnmber q a positive result, we shall have

$$\text{for} \left| \begin{matrix} x = p \\ x = q \end{matrix} \right| \text{ the inequalities } \left| \begin{matrix} P \asymp N \\ P \asymp N \end{matrix} \right|$$

And as each of these functions contains only whole positive powers of x, it is plain, that P and N will continually increase as x increases.

But P being less than N, for $x = p$, the increase of P, for any augmented value of x, must neces-

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(x) In order to avoid all unnecessary trials in this method of proceeding, it will be sufficient, according to what has been before shown, to take for the value of the limit e the greatest negative coefficient of the proposed equation, augmented by 1, and then making it less and less till it gives the least positive result that can be obtained.

sarily be greater than that of N, or otherwise P − N could never be = 0.

If, therefore, x be supposed to increase uniformly, by insensible degrees, from p to q, P, which, at first, was less than N, will, at length, become greater than N, when $x = q$.

Whence there must be some intermediate value of x, between the limits p and q, that will render

$$P = N, \text{ or } P - N = 0;$$

which value, whether known or unknown, must, evidently, be one of the real roots of the equation.

Thus, for example, since the hypothesis $x = 3$ and $x = 5$ reduces the polynomial

$$x^3 - 2x^2 + 3x - 44 = 0$$

to -26 and $+6$, there must necessarily exist a value of x between 3 and 5, that will reduce the same expression to 0; which is in effect $+4$, as will be found by trial.

The same principle might also be shown to take place when the numbers p and q are both negative, or one positive and the other negative; but the case here treated of will be found sufficient to answer all the purposes intended by the proposition (y).

It may likewise be easily proved, that if two

(y) The two polynomials, P and N, in the above demonstration, may be assimilated, according to the observation made by Lagrange, to two moving bodies, which set out at the same time from different points, and proceed in the same direction; in which case, if the hindmost moves faster than the foremost, it will, at length, pass by it; and consequently, in some part of the time, they must have been exactly together.

magnitudes, when substituted for the unknown quantity, give results with contrary signs, three, or some other odd number of roots, may sometimes lie between them.

And if the results arising from the substitution of two magnitudes for the unknown quantity, be both positive, or both negative, either no root of the equation, or an even number of roots, must always lie between them.

Hence, it must not be hastily concluded, that because two substitutions for the unknown quantity give results of the same kind, that no root of the equation can lie between them; or that two substitutions, which give results with contrary signs, can have only one real root lying between them (z).

14. Every equation, of an odd number of dimensions, has, at least, one real root, of a contrary sign to that of its last term; and every equation of an even number of dimensions, which has its last term negative, has, at least, two real roots, with contrary signs.

For let there be assumed, in the first place, the equation

$$x^{2m+1} + \text{A}x^{2m} + \text{B}x^{2m-1} + \ldots + \text{T}x - \text{V} = 0 \ldots (1)$$

(z) From what has been shown in the above proposition, it is plain, that if any series of quantities can be found, which give as many results, that are alternately positive and negative, as the equation has dimensions, they will be limits of the several roots of that equation; for since there are as many terms as the equation has dimensions, and an odd number of roots always lies between each two succeeding terms of the series, this number, in the case here mentioned, cannot exceed one.

of an odd number of dimensions; in which the last term is $-$, and the other terms either $+$ or $-$.

Then, if x be taken equal to 0, the result will evidently be negative, being $= -\text{V}$.

And supposing N to be the greatest negative coefficient of the equation, the substitution of $\text{N}+1$ for x, will give a positive result (Art. 12).

Whence, by the last proposition, there must necessarily be some real positive value of x, lying between 0 and $\text{N}+1$.

Also, let there be taken, in the second place, the equation

$$x^{2m+1}+\text{A}x^{2m}+\text{B}x^{2m-1}+\ .\ .\ .\ +\text{T}x+\text{V}=0\ .\ .\ .\ (2)$$

in which the last term is positive, and the other terms $+$ or $-$, as before.

Then, if $-x$ be substituted for $+x$, the given equation will be transformed to

$$-x^{2m+1}+\text{A}x^{2m}-\text{B}x^{2m-1}+\ .\ .\ .\ -\text{T}x+\text{V}=0.$$

And by taking all the terms to the other side of the equation, or, which is the same thing, changing their signs, we shall have

$$x^{2m+1}-\text{A}x^{2m}+\text{B}x^{2m-1}-\ .\ .\ .\ +\text{T}x-\text{V}=0.$$

Whence, as this last equation is now of the same form as equation (1), it must, also, have a real positive root.

And since $-x$ was barely substituted for $+x$, in equation (2), this equation must necessarily have a real negative root.

Again, for an equation of an even number of dimensions, let

$$x^{2m}+\text{A}x^{2m-1}+\text{B}x^{2m-2}+\ .\ .\ .\ +\text{T}x-\text{V}=0.\ .\ .\ .\ (3)$$

Then, if x be taken $=0$, as above, the result will evidently be negative; being, as in the first case, $= -V$.

And supposing N to be the greatest negative coefficient, the substitution of $N+1$ for x will give a positive result.

Whence there is also, in this case, some real positive root, or value of x, lying between 0 and $N+1$.

Also, if $-x$ be substituted for $+x$, in this last equation, it will become of the form

$$x^{2m} - Ax^{2m-1} + Bx^{2m-2} - \ldots\ldots - Tx - V = 0,$$

where, the last term being negative, it must necessarily have one positive root, as before.

And as $-x$, in this case, is the same as $+x$ in equation (3), that equation must also have one real negative root.

Whence equation (3) has, at least, two real roots, one of which is positive and the other negative, as was to be shown.

In the case where an equation of an even number of dimensions has its last term positive, the substitutions above employed will give two results with the same sign; so that it cannot thence be concluded whether the equation has any real root or not.

15. Imaginary quantities, which are in every case of the forms $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$, always enter an equation by pairs.

For if $a+b\sqrt{-1}$, taken as one of the roots, be substituted for x in the equation

$$x^m + Ax^{m-1} + Bx^{m-2} + \ldots\ldots + Tx + V = 0,$$

the result will be composed of two parts; one of which consists of real quantities that involve the powers of a and the even powers of $b\sqrt{-1}$, and the other of imaginary quantities that involve the odd powers of $b\sqrt{-1}$.

Let, therefore, the sum of the real quantities be denoted by P, and that of the imaginary quantities by Q; then $P+Q$ will be the whole result.

Again, if $a-b\sqrt{-1}$ be substituted for x, the real part of the result will still be the same as before; and the imaginary part, which arises from the odd powers of $b\sqrt{-1}$, will only differ from the former imaginary part by its sign; so that the result, in this case, will be $P-Q$.

But since $a+b\sqrt{-1}$ is, by hypothesis, a root of the equation, $P+Q$ must be $=0$; and, as no part of P can be destroyed by Q, P and Q will be each $=0$.

Whence, also, $P-Q=0$; that is, the result arising from the substitution of $a-b\sqrt{-1}$ for x is 0; and consequently both $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$ will be roots of the equation.

But this conclusion will not take place if the equation be supposed to have an odd number of imaginary roots.

Hence, as imaginary quantities always enter into equations by pairs, an equation of an odd number of dimensions cannot have an even number of real roots, and an equation of an even number of dimensions cannot have an odd number of real roots.

16. When the two members of any identity consist of the same successive powers of some inde-

terminate quantity x, the coefficients of all the like powers of x, in that identity, will be equal to each other.

For every expression of this kind constitutes an equality, which ought to be true for any value whatever that can be given to x (def. Part I).

Whence, if the coefficients of the same powers of x were not equal, the identity, by taking all its terms to the same side, would become an equation, which can have only as many values of x as there are units in the index of its highest power.

And, therefore, as this limitation of the roots is contrary to the nature of an identity, it follows, that the coefficients of all the like powers of x, in each of its members, must be equal.

Thus, for example, if there be taken the identity

$$\text{A} + \text{B}x + \text{C}x^2 = a + bx + cx^2,$$

we shall necessarily have, from the nature of the expression,

$$\text{A} = a, \ \text{B} = b, \text{ and } \text{C} = c.$$

Or, otherwise, by bringing the terms to the same side, there would arise the quadratic

$$(\text{C} - c)x^2 + (\text{B} - b)x + \text{A} - a = 0,$$

where x cannot have more than two values; which is contrary to the hypothesis.

SIMPLE EQUATIONS.

(9) The rules for resolving simple equations, that contain only one unknown quantity, are founded upon such evident principles, that nothing more need be said on this head than what has been

already observed in the corresponding article (o) of the former part of this work (a).

But in the case where two or more simple equations are proposed, involving the same number of unknown quantities, the question becomes more complicated; and, on that account, may appear to require some farther elucidation.

In addition, therefore, to the modes of solution pointed out in the practical rules before given, it will be here proper to observe, that any two equations of the form

$$ax + by = c,$$
$$a'x + b'y = c',$$

may be readily reduced to a more commodious state, by multiplying the first of them by some indeterminate quantity m, and then subtracting the second from this result; when there will arise

$$(am - a')x + (bm - b')y = cm - c'.$$

And as the value of m, in this equation, is not fixed, we can take $bm = b'$, or $m = \frac{b'}{b}$; in which case the second term will disappear, and we shall have

$$x = \frac{cm - c'}{am - a'} = \frac{cb' - bc'}{ab' - ba'}.$$

(a) It may here be remarked, as in the case of simple equations above referred to, that an inequality will not be altered, by performing the same operation on each of its two members.

Thus, if $2x + a \lesseqgtr b$, we shall have $2x + a + c \lesseqgtr b + c$; $2x + a - c \lesseqgtr b - c$; $c(2x + a) \lesseqgtr cb$; $(2x + a)^m \lesseqgtr b^m$.

And if $5x - a \lesseqgtr b$, we shall have $5x - a + c \lesseqgtr b + c$; $5x - a - c \lesseqgtr b - c$; $\sqrt[n]{(5x - a)} \lesseqgtr \sqrt[n]{b}$, &c.

Also, as x is now determined, we can take $am = a'$, or $m = \frac{a'}{a}$; according to which supposition the first term will disappear, and the result will give

$$y = \frac{cm - c'}{bm - b'} = \frac{ac' - ca'}{ab' - ba'}.$$

Which proceeding has the advantage of greater brevity than that in common use, as we can deduce the values of each of the unknown quantities x and y from the same equation.

A similar method of solution may likewise be readily applied to any three equations of the first degree, containing three unknown quantities, as

$$ax + by + cz = d,$$
$$a'x + b'y + c'z = d',$$
$$a''x + b''y + c''z = d''.$$

For, if the first of these be multiplied by m, and the second by n, we shall have, by subtracting the third from their sum,

$$(am + a'n - a'')x + (bm + b'n - b'')y + (cm + c'n - c'')z$$
$$= dm + d'n - d''.$$

Whence, if there be taken $am + a'n - a'' = 0$ and $bm + b'n - b'' = 0$, there will remain $(cm + c'n - c'')z = dm + d'n - d''$; which gives

$$z = \frac{dm + d'n - d''}{cm + c'n - c''}.$$

Or since, from the two equations in m and n, there arises

$$m = \frac{a''b' - b''a'}{ab' - ba'}, \quad n = \frac{ab'' - ba''}{ab' - ba'},$$

we shall readily obtain, by the substitution of these values,

$$z = \frac{d(b'a'' - a'b'') + d'(ab'' - ba'') + d''(ba' - ab')}{c(b'a'' - a'b'') + c'(ab'' - ba'') + d''(ba' - ab')}.$$

Also, by taking $am + a'n - a'' = 0$ and $cm + c'n - c'' = 0$, we shall have, by operating as before,

$$y = \frac{d(c'a'' - a'c'') + d'(ac'' - ca'') + d''(ca' - ac')}{b(c'a'' - a'c'') + b'(ac'' - ca'') + b''(ca' - ac')}.$$

And, by again taking $bm + b'n - b'' = 0$ and $cm' + c'n - c'' = 0$, the result, obtained in the same manner, will give

$$x = \frac{d(c'b'' - b'c'') + d'(bc'' - cb'') + d''(cb' - bc')}{a(c'b'' - b'c'') + a'(bc'' - cb'') + a''(cb' - bc')}.$$

Which results are similar to those given for the values of x, y, and z, in the note to the practical rule, Part 1 (*b*).

And in the same way may this analysis be applied to four, five, or more, simple equations, containing as many unknown quantities; but as these seldom occur in practice, it would be unnecessary to pursue the subject to a greater length.

QUADRATIC EQUATIONS.

(P) THE general method of resolving quadratic equations having been already sufficiently explained, it will here only be necessary to point out some particulars relating to the nature of their roots, and the signs of the terms, which could not be conveniently introduced into the practical part of the work.

(*b*) It may here be observed, that the values of x, y, and z, in this case, might have been obtained by eliminating x in the first and second equations, according to the process pointed out in the rule, Part 1; but the equation in z, after eliminating y in the resulting equations, would then have had a factor too much in all its terms, owing to the given equations not being equally employed in the operation.

For this purpose, therefore, let there be taken the four following general formulæ:

1. $x^2+ax-b=0$ . . where . . $x=-\frac{a}{2}\pm\sqrt{(\frac{1}{4}a^2+b)}$

2. $x^2-ax-b=0$ . . . . . $x=\frac{a}{2}\pm\sqrt{(\frac{1}{4}a^2+b)}$

3. $x^2-ax+b=0$ . . . . . $x=\frac{a}{2}\pm\sqrt{(\frac{1}{4}a^2-b)}$

4. $x^2+ax+b=0$ . . . . . $x=-\frac{a}{2}\pm\sqrt{(\frac{1}{4}a^2-b)}$

which comprehend all the roots and changes of the signs that can arise in any equation of this kind.

Then it may be observed, with respect to these forms, that

1. In the case $x^2+ax-b=0$, where $x=-\frac{1}{2}a+\sqrt{(\frac{1}{4}a^2+b)}$, or $-\frac{1}{2}a-\sqrt{(\frac{1}{4}a^2+b)}$, the first, or least value of x, abstracting from its sign, must be positive, because $\sqrt{(\frac{1}{4}a^2+b)}$ is greater than $\sqrt{\frac{1}{4}a^2}$, or its equal $\frac{1}{2}a$; and its second, or greatest value, will evidently be negative, because each of the terms, of which it is composed, is negative.

2. In the case $x^2-ax-b=0$, where $x=\frac{1}{2}a+\sqrt{(\frac{1}{4}a^2+b)}$, or $\frac{1}{2}a-\sqrt{(\frac{1}{4}a^2+b)}$, the first, or greatest value of x, is manifestly positive, being the sum of two positive terms; and the second, or least value, will be negative, because $\sqrt{(\frac{1}{4}a^2+b)}$ is greater than $\sqrt{\frac{1}{4}a^2}$, or its equal $\frac{1}{2}a$.

3. In the case $x^2-ax+b=0$, where $x=\frac{1}{2}a+\sqrt{(\frac{1}{4}a^2-b)}$, or $\frac{1}{2}a-\sqrt{(\frac{1}{4}a^2-b)}$, both the values of x will be positive, supposing $\frac{1}{4}a^2$ to be greater than b; for its first, or greatest value, is then evidently positive, being composed of two positive terms; and

its second, or least value, will also be positive; because $\sqrt{(\frac{1}{4}a^2 - b)}$ is less than $\sqrt{\frac{1}{4}a^2}$, or its equal $\frac{1}{2}a$.

But if $\frac{1}{4}a^2$, in this case, be less than b, the solution of the proposed equation will be impossible; because the quantity $\frac{1}{4}a^2 - b$, under the radical, is then negative; and consequently $\sqrt{(\frac{1}{4}a^2 - b)}$ will be imaginary, or of no assignable value.

Hence, in the first and second cases, $x^2 + ax - b = 0$ and $x^2 - ax - b = 0$, where there is one change of the signs from + to −, and one permanence +, +, or −, −, one of the values of x is always affirmative and the other negative; the positive value being the least of the two in the former, and the greatest in the latter (c).

And in the third case $x^2 - ax + b = 0$, where there are two changes of the signs, one from + to −,

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(c) The observation above made may be considered as a confirmation, in the partial instance here given, of the rule first laid down by Descartes, that no equation can have a greater number of positive roots than it has variations of the signs, from + to −, or from − to +, nor a greater number of negative roots than it has permanencies, or continuations of the same sign, from + to +, or from − to −; and if an equation contains only real roots, it will have exactly the same number of positive roots as it has variations of the signs, and the same number of negative roots as there are permanencies of the signs.

Thus, the equation $x^3 + x^2 - 10x + 8 = 0$, whose roots are all real, has two positive roots and one negative; because the signs are +, +, −, +, in which there are two variations, one from + to −, and the other from − to +, and one permanence + to +.

And if the roots of the equation $x^3 - ax - b = 0$ be real, two of them will be negative and the third positive; because, when it is put under the form $x^3 \pm 0 - ax - b = 0$, there will be two permanencies of the signs and one variation.

and the other from $-$ to $+$, both the values of x are affirmative, if $\frac{1}{4}a^2$ be greater than b; but if $\frac{1}{4}a^2$ be less than b, each of these roots will be imaginary, being of the form

$$a' + b'\sqrt{-1}, \text{ and } a' - b'\sqrt{-1}.$$

4. Also, in the case $x^2 + ax + b = 0$, where $x = -\frac{1}{2}a + \sqrt{(\frac{1}{4}a^2 - b)}$, or $-\frac{1}{2}a - \sqrt{(\frac{1}{4}a^2 - b)}$, the two values of x will be both negative, or both imaginary, according as $\frac{1}{4}a^2$ is greater or less than b; the imaginary roots, when they occur, being here of the form

$$-(a' + b'\sqrt{-1}) \text{ and } -(a' - b'\sqrt{-1}).$$

From which it follows, that if all the terms of a quadratic equation, when brought to the left hand side, be positive, its two roots will be both negative, or both imaginary; and, conversely, if each of the roots be negative, or each imaginary, the signs of all the terms will be positive.

So that of all quadratic equations, which can have any real positive root, that of the third form

$$x^2 - ax + b = 0,$$

is the only one, where the solution, for certain numeral values of a and b, will become impossible.

With respect to the second rule in this article, which treats of the method of determining the roots of these equations by means of the tables of sines and tangents, the formulæ commonly used for this purpose may be obtained as follows:

Let the given equation, as in the first case of the rule, be of the form

$$x^2 + ax = b.$$

Then the two roots, or values of x, when properly reduced, are

$$x = -\frac{a}{2}\{1 \mp \sqrt{(1 + \frac{4b}{a^2})}\}.$$

Whence, if there be taken $\frac{2\sqrt{b}}{a} = \tan. z$, we shall have $\frac{a}{2} = \frac{\sqrt{b}}{\tan. z} = \frac{\cos. z\sqrt{b}}{\sin. z}$; and, consequently,

$$x = -\frac{\cos. z\sqrt{b}}{\sin. z}\{1 \mp \sqrt{(1 + \tan.^2 z)}\} = -\frac{\cos. z\sqrt{b}}{\sin. z}(1 \mp \frac{1}{\cos. z})$$

$$= \pm \frac{1 \mp \cos. z}{\sin. z}\sqrt{b}\,(d).$$

But since $\frac{1 - \cos. z}{\sin. z} = \tan. \frac{1}{2}z$, and $\frac{1 + \cos. z}{\sin. z} = \cot. \frac{1}{2}z$, these expressions will become

$$x = \sqrt{b} \times \tan. \tfrac{1}{2}z;\ x = -\sqrt{b} \times \cot. \tfrac{1}{2}z.$$

Which are the two values of x given in the practical rule.

Again, if the given equation, as in the second case, be of the form

$$x^2 - ax = b,$$

its two roots, or the values of x, reduced as before, will be

$$x = \frac{a}{2}\{1 \pm \sqrt{(1 + \frac{4b}{a^2})}\}.$$

And, consequently, by putting $\frac{2\sqrt{b}}{a} = \tan. z$, and following the same process as above, we shall have

$$x = \sqrt{b} \times \cot. \tfrac{1}{2}z;\ x = -\sqrt{b} \times \tan. \tfrac{1}{2}z.$$

(*d*) For a copious and methodical collection of formulæ, of the kind here used, with their investigations, see my Treatise on Plane and Spherical Trigonometry, where the admirable improvements that have been made in the analytical branch of this science, by Euler, Delambre, and other celebrated foreigners, were first introduced to the notice of the English student.

Which are the two values of x in the second part of the rule.

Lastly, when the proposed equation is of the form

$$x^2 - ax = -b,$$

its two roots, or the values of x, reduced in a similar manner, are

$$x = \frac{a}{2}\{1 \pm \sqrt{(1 - \frac{4b}{a^2})}\}$$

Whence, putting $\frac{2\sqrt{b}}{a} = \text{sin. } z$, we shall have $\frac{a}{2} = \frac{\sqrt{b}}{\text{sin. } z}$; and, consequently,

$$x = \frac{\sqrt{b}}{\text{sin. } z}\{1 \pm \sqrt{(1 - \text{sin.}^2 z)}\} = \frac{\sqrt{b}}{\text{sin. } z}(1 \pm \text{cos. } z).$$

But since $\frac{1 + \text{cos. } z}{\text{sin. } z} = \text{cot. } \frac{1}{2}z$, and $\frac{1 - \text{cos. } z}{\text{sin. } z} = \text{tan. } \frac{1}{2}z$, the last expression will become

$$x = \sqrt{b} \times \text{cot. } \tfrac{1}{2}z\,;\ x = \sqrt{b} \times \text{tan. } \tfrac{1}{2}z.$$

In which case it is to be observed, that, if b be greater than $\frac{1}{4}a^2$, or $\frac{4b}{a^2} = 1$, the question will be impossible, as $\sqrt{(\frac{1}{4}a^2 - b)}$ is then imaginary, and consequently unassignable.

CUBIC EQUATIONS.

(a) Having given, in the former part of the present work, all the most useful practical rules that have hitherto been devised for determining the roots of cubic equations, I shall here subjoin their demonstrations; supplying, at the same time, such farther remarks and observations, as may best serve to elucidate some of the more curious particulars relating to this branch of the science.

For this purpose, let the equation that is to be resolved, when deprived of its second term, be represented under the form

$$x^3 + ax = b.$$

In which case, a and b may be any positive or negative numbers whatever (e).

Then, if there be taken $x = y + z$, the given equation, by substitution, will become

$$y^3 + 3y^2z + 3yz^2 + z^3 + ay + az = b.$$

Or, since $3y^2z + 3yz^2 = 3yz(y + z)$, and $ay + az = a(y + z)$, it will be

$$y^3 + z^3 + (3yz + a)(y + z) = b;$$

where, as both y and z are unknown quantities, the value of one of them may be taken at pleasure, provided the other be the complement of x.

Let us suppose, therefore, z to be of such a value, that

$$3yz + a = 0, \text{ or } z = -\frac{a}{3y}.$$

In which case there will evidently remain the equation

$$y^3 + z^3 = b.$$

Whence, by substituting $-\frac{a}{3y}$ for z, in this last expression, and reducing and transposing the terms, we shall have

$$y^6 - by^3 = \frac{a^3}{27}.$$

(e) The method of investigation here followed, is that which is generally attributed to Cardan, whose name the formula for the solution of cubic equations commonly bears; but Lagrange considers it as due to Hudde, a Dutch analyst, of considerable reputation, who flourished about the middle of the seventeenth century.

Which equation, being resolved after the manner of a quadratic, gives

$$y=\sqrt[3]{\{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}\}}$$

And, consequently, since $z=-\frac{a}{3y}$, we shall also have

$$z=\frac{-\frac{1}{3}a}{\sqrt[3]{[\frac{1}{2}b+\sqrt{(\frac{1}{4}b^2+\frac{1}{27}a^3)}]}}$$

Or, multiplying both the numerator and denominator of the right hand member by $\sqrt[3]{\{\frac{1}{2}-\sqrt{(\frac{1}{4}b^2+\frac{1}{27}a^3)}\}}$,

$$z=\sqrt[3]{\{\tfrac{1}{2}b-\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}\}}.$$

Hence, according to the supposition first made, we shall have

$$x=\sqrt[3]{\{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}\}}+\sqrt[3]{\{\tfrac{1}{2}b-\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}\}}.$$

Which expression, by taking a and b either in $+$ or in $-$, as they may happen to be, will give the first root of the proposed equation, for each of the cases laid down in the practical rule.

And since one value of x is now determined, the equation may be depressed to a quadratic, and the other two roots found, as in the note to the above mentioned rule.

Or, the two latter of these roots may be obtained in a different manner, as follows.

Let there be put, as in the case of the practical rule,

$$\sqrt[3]{\{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}\}}=\alpha,\ \sqrt[3]{\{\tfrac{1}{2}b-\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}\}}=\beta$$

Then we shall have, by the addition and multiplication of these two expressions,

$$x=\alpha+\beta, \text{ and } \alpha\beta=\tfrac{1}{3}a, \text{ or } 3\alpha\beta=a.$$

And, consequently, if $\alpha+\beta$ be substituted for x, in the original equation there will arise

$$(\alpha+\beta)^3+a(\alpha+\beta)=b.$$

Whence, equating this value of b with the former, and transposing the terms, we shall have

$$x^3-(\alpha+\beta)^3=a\{x-(\alpha+\beta)\}=-3\alpha\beta\{x-(\alpha+\beta)\}$$

And dividing the first and last members of this equation by $x-(\alpha+\beta)$ the result will be

$$x^2+(\alpha+\beta)x+(\alpha+\beta)^2=-3\alpha\beta.$$

Which last equation, when resolved in the usual way, gives

$$x=-\tfrac{1}{2}(\alpha+\beta)\pm\tfrac{1}{2}(\alpha-\beta)\sqrt{-3},$$

for the other two roots of the proposed equation, as was required.

Hence, if the values of α and β be now restored, the three roots of the equation

$$x^3+ax=\pm b,$$

putting r for the value of the first, and using the upper or under sign, according as b is positive or negative, will be as follows.

$$x=\sqrt[3]{\{\pm\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}\}}+\sqrt[3]{\{\pm\tfrac{1}{2}b-\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}\}} \quad . \; . \; . \; (r)$$

$$x=\mp\frac{r}{2}+\frac{1}{2}\{\sqrt[3]{[\pm\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}]}-\sqrt[3]{[\pm\tfrac{1}{2}b-\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}]}\}\sqrt{-3}$$

$$x=\mp\frac{r}{2}-\frac{1}{2}\{\sqrt[3]{[\pm\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}]}-\sqrt[3]{[\pm\tfrac{1}{2}b-\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}]}\}\sqrt{-3}$$

And if the coefficient a of the second term be $-$, in which case the equation is of the form

$$x^3 - ax = \pm b,$$

its three roots, putting r for the value of the first, and taking the signs as before, will be

$$x = \sqrt[3]{\{\pm\tfrac{1}{2}b + \sqrt{(\tfrac{1}{4}b^2 - \tfrac{1}{27}a^3)}\}} + \sqrt[3]{\{\pm\tfrac{1}{2}b - \sqrt{(\tfrac{1}{4}b^2 - \tfrac{1}{27}a^3)}\}} \quad . \; . \; . \; (r)$$

$$x = \mp\frac{r}{2} + \frac{1}{2}\{\sqrt[3]{[\pm\tfrac{1}{2}b + \sqrt{(\tfrac{1}{4}b^2 - \tfrac{1}{27}a^3)}]} - \sqrt[3]{[\pm\tfrac{1}{2}b - \sqrt{(\tfrac{1}{4}b^2 - \tfrac{1}{27}a^3)}]}\}\sqrt{-3}$$

$$x = \mp\frac{r}{2} - \frac{1}{2}\{\sqrt{[\pm\tfrac{1}{2}b + \sqrt{(\tfrac{1}{4}b^2 - \tfrac{1}{27}a^3)}]} - \sqrt[3]{[\pm\tfrac{1}{2}b - \sqrt{(\tfrac{1}{4}b^2 - \tfrac{1}{27}a^3)}]}\}\sqrt{-3}$$

But it is to be observed, that only the first of the three roots here given, for each of these equations, is real, except when $\frac{1}{27}a^3$, in the latter, is greater than $\frac{1}{4}b^2$; in which case, they are all real, though neither of them can be found from the above formulæ, on account of $\sqrt{(\frac{1}{4}b^2 - \frac{1}{27}a^3)}$ being imaginary.

This, therefore, has been usually called the *irreducible case* of cubic equations; because the common rule, which succeeds when there is only one real root, here fails; and no other formula, purely algebraical, has yet been discovered, by which either of the roots, in this case, can be determined.

It is proper to remark, however, that, although the compound expression, here given, for finding the value of x, in this case, consists of imaginary

surds, yet the whole expression will not be imaginary, as may be readily shown by converting it into a series, where the radical parts will disappear. The same will also take place, when the two branches of the formula, in any numeral example of this kind, are, each, complete cubes.

Thus, if it were required, for instance, to find one of the values of x in the equation

$$x^3 - 30x = 36,$$

we shall have, by substituting 30 for a, and 36 for b, in the first of the three last formulæ,

$$x = \sqrt[3]{(18 + 26\sqrt{-1})} + \sqrt[3]{(18 - 26\sqrt{-1})}$$

where it can be found, either by cubing, or from (1) Art. 8, that

$$\sqrt[3]{(18 + 26\sqrt{-1})} = 2 + \sqrt{-1},$$
$$\sqrt[3]{(18 - 26\sqrt{-1})} = 2 - \sqrt{-1}.$$

And, consequently, the sum of the two radicals $= 4$, which is the root sought.

Hence, some of the early analysts, who, in their attempts to resolve equations of the higher orders, observed the species of paradox above mentioned, endeavoured to free the expression from imaginary surds, by extracting the cube root of each of its two branches.

It was in this manner, says Lagrange, (*Séances des Ecoles Normales*, T. iii. p. 295), that Bombelli convinced himself of the reality of the imaginary expression, in the formula for the irreducible case; but as the extraction could not be always effected, except by means of series, he failed in obtaining a general demonstration of the proposition in question.

This defect, however, does not take place with

respect to square radicals, or those which have some exact power of 2 for their exponent.

As, for instance, if we have the compound quantity

$$\sqrt{(a+b\sqrt{-1})}+\sqrt{(a-b\sqrt{-1})},$$

which is composed of two imaginary radicals, its square will be

$$2a+2\sqrt{(a^2+b^2)},$$

which is a quantity necessarily positive.

And, consequently, by extracting its square root, we shall have

$$\sqrt{\{2a+2\sqrt{(a^2+b^2)}\}}$$

for the real value of the quantity proposed.

But if, instead of the sum, we had the difference of the same two radicals, its square would be

$$2a-2\sqrt{(a^2+b^2)},$$

which is a quantity necessarily negative.

And, consequently, by taking its square root, we shall have the simple imaginary expression,

$$\sqrt{\{2a-2\sqrt{(a^2+b^2)}\}}.$$

Again, if the compound quantity proposed be

$$\sqrt[4]{(a+b\sqrt{-1})}+\sqrt[4]{(a-b\sqrt{-1})},$$

we shall have, by elevating it to the second power,

$$\sqrt{\{2a+2\sqrt{(a^2+b^2)}\}}+2\sqrt[4]{(a^2+b^2)},$$

which is a real positive quantity.

Whence, extracting its square root, we shall have a real value of the quantity proposed; and so on, for other even roots.

But if we would apply this method to cubic radicals, we shall again fall upon an equation of the third degree, in the irreducible case.

For let there be taken the compound quantity

$$x=\sqrt[3]{(a+b\sqrt{-1})}+\sqrt[3]{(a-b\sqrt{-1})}.$$

Then we shall have, by raising it to the third power,

$$x^3 = 2a + 3\sqrt[3]{(a^2+b^2)}\{\sqrt[3]{(a+b\sqrt{-1})} + \sqrt[3]{(a-b\sqrt{-1})}\}.$$

And, by substitution and transposition,

$$x^3 - 3\{\sqrt[3]{(a^2+b^2)}\}x = 2a,$$

which is a general formula for the irreducible case; because

$$\tfrac{1}{4}(2a)^2 - \tfrac{1}{27}\{3\sqrt[3]{(a^2+b^2)}\}^3 = -b^2.$$

And if b, in the proposed expression, be taken $=0$, we shall have

$$x = 2\sqrt[3]{a}.$$

It therefore remains to be proved, that when b has any real value, x will also have a corresponding real value.

But from the preceding equation it appears, that

$$\sqrt[3]{(a^2+b^2)} = \frac{x^3-2a}{3x}.$$

And, consequently, by cubing each of its sides, we shall have

$$a^2+b^2 = \frac{x^9-6ax^6+12a^2x^3-8a^3}{27x^3}.$$

Or, by taking a^2 to the right hand side of the equation,

$$b^2 = \frac{x^9-6ax^6-15a^2x^3-8a^3}{27x^3}.$$

Which expression may be put under the form

$$b^2 = \frac{1}{27}\left(1-\frac{8a}{x^3}\right)(x^3+a)^2.$$

Where it appears that $b=0$ when $x^3=8a$; and that afterwards b augments, without interruption, as x augments: for the factor $(x^3+a)^2$, in that case,

increases continually, as does also $(1-\frac{8a}{x^3})$; because as the denominator x^3 augments, the negative part $\frac{8a}{x^3}$, which is at first $=1$, becomes always less than 1.

Thus then, in augmenting, by insensible degrees, the value of x^3, from $8a$ to infinity, the value of b^2 also augments by insensible and corresponding degrees from 0 to infinity.

And therefore, reversedly, for every value of b^2, from 0 to infinity, there is a corresponding value of x^3, comprised between $8a$ and infinity; and as this is the case for every possible value of a, we may safely conclude, that whatever may be the values of a and b, the corresponding value of x^3, and consequently that of x, will always be real.

But, notwithstanding the reality of the root is thus confirmed, it does not appear that it can be represented in any other way, than by the imaginary expression

$$x=\sqrt[3]{\{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2-\tfrac{1}{27}a^3)}\}}+\sqrt[3]{\{\tfrac{1}{2}b-\sqrt{(\tfrac{1}{4}b^2-\tfrac{1}{27}a^3)}\}},$$

or by the series which arises from its development (*f*).

(*f*) It may here be worth while to observe, that several cubic equations that fall under the irreducible case, and which cannot therefore be resolved by Cardan's rule, may yet have their roots completely ascertained by other means, which are purely algebraical.

Thus, for example, in the equation $x^3-6x=1+3\sqrt{2}$, where $(\frac{6}{3})^3$, or 8, is greater than $(\frac{1+3\sqrt{2}}{2})^2$, it may be shown that

We must, therefore, regard such imaginary expressions, that answer to real quantities, as forming a new class of algebraical formulæ, which, although they have not, like others, the advantage of being estimated in numbers, in their usual state, have yet that, which is the only necessary quality in algebraical operations, of being employed as if they were not imaginary.

It may here likewise be farther shown, that when $\frac{1}{27}a^3$ is equal to, or greater than $\frac{1}{4}a^2$, the other two roots of the equation

$$x^3 - ax - b = 0,$$

as well as the first, will also be real quantities.

For suppose r, according to what has been proved above, to be one of the real roots of this equation; then

$$r^3 - ar - b = 0, \text{ or}$$
$$b = r^3 - ar.$$

Whence, by substitution, we shall have

$$x^3 - r^3 - a(x - r) = 0.$$

And, consequently, by dividing by $x - r$,

$$x^2 + rx + r^2 - a = 0, \text{ or}$$
$$x^2 + rx = a - r^2.$$

Which last equation contains the other two roots of the proposed equation, the solution of it giving

$$x = -\frac{r}{2} \pm \sqrt{(a - \frac{3}{4}r^2)}.$$

$x = 1 - \sqrt{3}/4 - \sqrt{3}/2$; though it is not easy to perceive how the knowledge of this circumstance can be turned to any account in the general solution of cubics.

Where, in order that each of these roots may be real, we must have

$$a = \text{ or } \pm \tfrac{3}{4}r^2, \text{ or } r = \text{ or } \pm 2\sqrt{\tfrac{1}{3}a}.$$

But if, in the second member of the equation,

$$b = r^3 - ar,$$

there be put, in the place of r, the equal, or less quantity $2\sqrt{\frac{1}{3}a}$, we shall also have

$$b = \text{ or } \pm (\tfrac{8}{3}a\sqrt{\tfrac{1}{3}a} - 2a\sqrt{\tfrac{1}{3}a}).$$

Or, bringing the terms to the same denominator, and taking their difference,

$$b = \text{ or } \pm \tfrac{2a}{3}\sqrt{\tfrac{1}{3}a}.$$

And consequently, by squaring each side, and dividing by 4, we have

$$\tfrac{1}{4}b^2 = \text{ or } \pm \tfrac{1}{27}a^3.$$

Whence it appears, that, if $\frac{1}{4}b^2 =$ or $\pm \frac{1}{27}a^3$, the three roots of the proposed equation will be all real; and if $\frac{1}{4}b^2 \pm \frac{1}{27}a^3$, one root only is real and two imaginary.

Having said thus much on this part of the subject, it will now be proper to give the investigations of the rules for resolving cubic equations by means of the tables of sines and tangents (*g*).

(*g*) The first hint respecting the solution of cubic equations, by means of the trisection of an angle, appears to have been given by Bombelli, in his Treatise of Algebra, printed at Bologna, 1579; but this method, by using the analytical instead of the

For which purpose let there be taken, as in the first case of the practical rule, the equation

$$x^3 + ax = b,$$

where the root, as found by the usual method, is

$$x = \sqrt[3]{\{\tfrac{1}{2}b + \sqrt{(\tfrac{1}{4}b^2 + \tfrac{1}{27}a^3)}\}} + \sqrt[3]{\{\tfrac{1}{2}b - \sqrt{(\tfrac{1}{4}b^2 + \tfrac{1}{27}a^3)}\}}.$$

Or, when reduced to a form more suitable to the present investigation,

$$x = \sqrt[3]{\{\tfrac{1}{2}b + \tfrac{1}{2}b\sqrt{(1 + \tfrac{4a^3}{27b^2})}\}} + \sqrt[3]{\{\tfrac{1}{2}b - \tfrac{1}{2}b\sqrt{(1 + \tfrac{4a^3}{27b^2})}\}}.$$

Then, by taking $\frac{4a^3}{27b^2} = \tan.^2 z$, or, which is the same thing,

$$\tan. z = \frac{2a}{3b}\sqrt{\frac{a}{3}}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we shall have, by substitution, and the common trigonometrical formulæ used in the last article,

$$x = \sqrt[3]{\{\tfrac{1}{2}b(1 + \tfrac{1}{\cos. z})\}} + \sqrt[3]{\{\tfrac{1}{2}b(1 - \tfrac{1}{\cos. z})\}}.$$

Or, since $\frac{b}{2} = \frac{a}{3\tan. z}\sqrt{\frac{a}{3}}$, $1 + \frac{1}{\cos. z} = \frac{1 + \cos. z}{\cos. z}$, and $1 - \frac{1}{\cos. z} = -\frac{1 - \cos. z}{\cos. z}$, this expression will become

$$x = \sqrt[3]{\{\tfrac{a}{3}\sqrt{\tfrac{a}{3}}(\tfrac{1 + \cos. z}{\cos. z \tan. z})\}} - \sqrt[3]{\{\tfrac{a}{3}\sqrt{\tfrac{a}{3}}(\tfrac{1 - \cos. z}{\cos. z \tan. z})\}}.$$

But $\sqrt[3]{(\frac{a}{3}\sqrt{\frac{a}{3}})} = \sqrt{\frac{a}{3}}$, $\frac{1 + \cos. z}{\cos. z \tan. z} = \frac{1 + \cos. z}{\sin. z} = \cot. \frac{1}{2}z$, and $\frac{1 - \cos. z}{\cos. z \tan. z} = \frac{1 - \cos. z}{\sin. z} = \tan. \frac{1}{2}z$; whence

$$x = \sqrt{\frac{a}{3}}(\sqrt[3]{\cot. \tfrac{1}{2}z} - \sqrt[3]{\tan. \tfrac{1}{2}z}).$$

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geometrical form of investigation, has since been greatly improved, and will, it is hoped, be here found as simple and perspicuous as the nature of the subject will admit.

Or, putting $\sqrt[3]{\cot.\frac{1}{2}z} = \cot.u$; which gives, at the same time,

$$\sqrt[3]{\tan.\tfrac{1}{2}z} = \tan.u, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

we shall have, by the trigonometrical formulæ before referred to,

$$x = \sqrt{\frac{a}{3}}(\cot.u - \tan.u).$$

Hence, also, because $\cot.u - \tan.u = 2\cot.2u$, the first result will be

$$x = 2\sqrt{\frac{a}{3}} \times \cot.2u, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where the equations 1, 2, 3, by taking the radius $= 1$, are the same as those given in the practical rule.

Again, let the proposed equation be, as in the second case, of the form

$$x^3 - ax = b;$$

where $\frac{1}{4}b^2$ is supposed to be greater than $\frac{1}{27}a^3$, or $27b^2 = 4a^3$.

Then the root, found by the common rule, as before, will be

$$x = \sqrt[3]{\{\tfrac{1}{2}b + \tfrac{1}{2}b\sqrt{(1 - \tfrac{4a^3}{27b^2})}\}} + \sqrt[3]{\{\tfrac{1}{2}b - \tfrac{1}{2}b\sqrt{(1 - \tfrac{4a^3}{27b^2})}\}}.$$

And by taking $\frac{4a^3}{27b^2} = \sin.^2 z$, or, which is the same thing,

$$\sin.z = \frac{2a}{3b}\sqrt{\frac{a}{3}}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we shall have, by transposition, and the common trigonometrical formulæ,

$$x = \sqrt[3]{[\tfrac{1}{2}b(1 + \cos.z)]} + \sqrt[3]{[\tfrac{1}{2}b(1 - \cos.z)]}.$$

Or, because $\frac{b}{2} = \frac{a}{3 \sin. z}\sqrt{\frac{a}{3}}$, as appears from the preceding formula, this last expression will become

$$x = \sqrt[3]{\left[\frac{a}{3}\sqrt{\frac{a}{3}}\left(\frac{1 + \cos. z}{\sin. z}\right)\right]} + \sqrt[3]{\left[\frac{a}{3}\sqrt{\frac{a}{3}}\left(\frac{1 - \cos. z}{\sin. z}\right)\right]}.$$

But $\sqrt[3]{\left(\frac{a}{3}\sqrt{\frac{a}{3}}\right)} = \sqrt{\frac{a}{3}}$, $\frac{1 + \cos. z}{\sin. z} = \cot. \frac{1}{2}z$, and $\frac{1 - \cos. z}{\sin. z} = \tan. \frac{1}{2}z$; whence

$$x = \sqrt{\frac{a}{3}}\left(\sqrt[3]{\cot. \tfrac{1}{2}z} + \sqrt[3]{\tan. \tfrac{1}{2}z}\right).$$

Or, putting $\sqrt[3]{\cot. \frac{1}{2}z} = \cot. u$, which gives, at the same time,

$$\sqrt[3]{\tan. \tfrac{1}{2}z} = \tan. u, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

we shall have, by the same trigonometrical formulæ, as before,

$$x = \sqrt{\frac{a}{3}}(\cot. u + \tan. u).$$

Hence, because $\cot. u + \tan. u = \frac{2}{\sin. 2u} = 2 \operatorname{cosec}. 2u$, the result will give

$$x = \sqrt{\frac{a}{3}} \times \frac{2}{\sin. 2u} = 2\sqrt{\frac{a}{3}} \times \operatorname{cosec}. 2u. \quad . \quad . \quad . \quad (3)$$

And if the absolute term b, in this case, had been negative, or $x^3 - ax = -b$, the result would then be

$$x = -\sqrt{\frac{a}{3}} \times \frac{2}{\sin. 2u} = -2\sqrt{\frac{a}{3}} \times \operatorname{cosec}. 2u;$$

where the equations 1, 2, 3, taking, as before, the radius $= 1$, are the same as those given in the rule.

In the third, or irreducible case, where the equation is still of the form

$$x^3 - ax = b;$$

but having $27b^2 = 4a^3$, another mode of solution

must be adopted, on account of $\sqrt{(\frac{1}{4}b^2-\frac{1}{27}a^3)}$ being imaginary

For this purpose, therefore, it is necessary to observe, that if any arc of a circle be denoted by z, we shall have, by trigonometry,

$$4\cos.^3 z-3\cos. z=\cos. 3z.$$

Or, by substituting $\frac{1}{3}z$ in the place of z, in each of the terms,

$$4\cos.^3\frac{z}{3}-3\cos.\frac{z}{3}=\cos. z. \quad . \quad . \quad . \quad (1)$$

Whence, in order to reduce the given equation $x^3-ax=b$ to this form, let there be taken

$$x=\frac{1}{y}\cos.\frac{z}{3}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Then, by substitution, there will arise $\frac{1}{y^3}\cos.^3\frac{z}{3}-\frac{a}{y}\cos.\frac{z}{3}=b$; or, multiplying by $4y^3$,

$$4\cos.^3\frac{z}{3}-4ay^2\cos.\frac{z}{3}=4by^3.$$

And, consequently, putting $4ay^2=3$, or $y=\frac{1}{2}\sqrt{\frac{3}{a}}$, this last equation will become

$$4\cos.^3\frac{z}{3}-3\cos.\frac{z}{3}=\frac{3b}{2a}\sqrt{\frac{3}{a}}.$$

Whence, as this is now similar to equation 1, we shall have, by equating the two right hand members,

$$\cos. z=\frac{3b}{2a}\sqrt{\frac{3}{a}}. \quad . \quad . \quad . \quad . \quad (3)$$

And, therefore, by substituting $\frac{1}{2}\sqrt{\frac{3}{a}}$ for y, in equation 2,

$$x=2\sqrt{\frac{a}{3}}\times\cos.\frac{z}{3}. \quad . \quad . \quad . \quad . \quad (4)$$

Which two last formulæ will enable us to determine one of the roots of the proposed equation, according to the directions given in the practical rule.

And since it is known from trigonometry, that any arc of a circle, increased by one or more circumferences, will have the same sign, cosine, &c. as before, it follows, that not only cos. z, but also

$$\cos.\,(360^\circ + z), \text{ and } \cos.\,(720 + z) = \frac{3b}{2a}\sqrt{\frac{3}{a}}.$$

Whence, by taking $\frac{1}{3}$ of each of these last expressions, and then finding the supplements of the results, we shall have

$$x = +2\sqrt{\frac{a}{3}} \times \cos.\,\frac{z}{3}$$

$$x = -2\sqrt{\frac{a}{3}} \times \cos.\,(60^\circ - \frac{z}{3})$$

$$x = -2\sqrt{\frac{a}{3}} \times \cos.\,(60^\circ + \frac{z}{3})$$

Which are the three roots of the proposed equation (*h*).

And if the absolute term b, in this case, be negative, the three roots of the equation

$$x^3 - ax = -b$$

will be the negatives of the former; *viz.*

(*h*) It may here be remarked, that if more than two circumferences of the circle had been added to the arc z, in this case, we should have fallen again upon the same values of x as those before found; which circumstance serves to confirm the common theory of equations, as far as regards those of the third degree; since these, as has been already shown, can never have more than three roots.

$$x = -2\sqrt{\frac{a}{3}} \times \cos.\frac{z}{3}$$

$$x = +2\sqrt{\frac{a}{3}} \times \cos.(60^\circ - \frac{z}{3})$$

$$x = +2\sqrt{\frac{a}{3}} \times \cos.(60^\circ + \frac{z}{3})$$

In each of which cases, $\cos. z = \frac{3b}{2a}\sqrt{\frac{3}{a}}$, as above shown.

With respect to the determination of the roots of cubic equations, by means of series, let there be given, as in the first case, the equation (*i*)

$$x^3 + ax = b.$$

Where the root, by transposing the terms of each of the two branches of the common formula, is

$$x = \sqrt[3]{\{\sqrt{(\tfrac{1}{4}b^2 + \tfrac{1}{27}a^3)} + \tfrac{1}{2}b\}} - \sqrt[3]{\{\sqrt{(\tfrac{1}{4}b^2 + \tfrac{1}{27}a^3)} - \tfrac{1}{2}b\}};$$

Or, by putting, for the sake of greater simplicity, $\sqrt{(\frac{1}{4}b^2 + \frac{1}{27}a^3)} = s$, and reducing the expression,

$$x = s^{\frac{1}{3}}\{\sqrt[3]{(1 + \frac{b}{2s})} - \sqrt[3]{(1 - \frac{b}{2s})}\}$$

Hence, extracting the roots of the right hand member of this equation, by the binomial theorem, there will arise

$$\sqrt[3]{(1 + \frac{b}{2s})} = 1 + \frac{1}{3}(\frac{b}{2s}) - \frac{2}{3.6}(\frac{b}{2s})^2 + \frac{2.5}{3.6.9}(\frac{b}{2s})^3 - \frac{2.5.8}{3.6.9.12}(\frac{b}{2s})^4 \&c.$$

$$\sqrt[3]{(1 - \frac{b}{2s})} = 1 - \frac{1}{3}(\frac{b}{2s}) - \frac{2}{3.6}(\frac{b}{2s})^2 - \frac{2.5}{3.6.9}(\frac{b}{2s})^3 - \frac{2.5.8}{3.6.9.12}(\frac{b}{2s})^4 \&c.$$

(*i*) This method of solving Cubic Equations, by means of series, was first given by Nicole, in the Memoirs of the Academy of Sciences, an. 1738, p. 99; and, afterwards, at greater length, by Clairaut, in his *Elémens d'Algébre*, 1746; but it is presumed that the investigation, given above, will be found far more perspicuous and complete, than the one here mentioned.

And, consequently, if the latter of these two series be subtracted from the former, the result, by making the first term of the remainder a multiplier, will give

$$x=\frac{2bs^{\frac{1}{3}}}{6s}\{1+\frac{2.5}{6.9}(\frac{b}{2s})^2+\frac{2.5.8.11}{6.9.12.15}(\frac{b}{2s})^4+$$
$$\frac{2.5.8.11.14\ 17}{6.9.12.15.18.21}(\frac{b}{2s})^6\&c.\}$$

But since $s=\sqrt{(\frac{1}{4}b^2+\frac{1}{27}a^3)}$, we shall have

$$(\frac{b}{2s})^2=\frac{27b^2}{27b^2+4a^3},\quad (\frac{b}{2s})^4=(\frac{27b^2}{27b^2+4a^3})^2\&c.$$

$$\frac{2bs^{\frac{1}{3}}}{6s}=\frac{2b}{6s^{\frac{2}{3}}}=\frac{2b}{\sqrt[3]{[2(27b^2+4a^3)]}}$$

Whence, by substitution,

$$x=\frac{2b}{\sqrt[3]{[2(27b^2+4a^3)]}}\{1+\frac{2.5}{6.9}(\frac{27b^2}{27b^2+4a^3})+\frac{2.5.8.11}{6.9.12.15}$$
$$(\frac{27b^2}{27b^2+4a^3})^2+\frac{2.5.8.11.14.17}{6.9.12.15.18.21}(\frac{27b^2}{27b^2+4a^3})^3\&c.\}$$

Which is the same expression as that given for the value of x in the practical rule.

Again, let the proposed equation be, as in the second case, of the form

$$x^3-ax=\pm b,$$

where $\frac{1}{4}b^2$ is supposed to be greater than $\frac{1}{27}a^3$, or $27b^2>4a^3$;

Then the root, as found by the common formula, when properly reduced, is

$$x=$$
$$\pm\sqrt[3]{\frac{b}{2}}\{\sqrt[3]{[1+\frac{2}{b}\sqrt{(\frac{1}{4}b^2-\frac{1}{27}a^3)}]}+\sqrt[3]{[1-\frac{2}{b}\sqrt{(\frac{1}{4}b^2-\frac{1}{27}a^3)}]}\}$$

Or, putting, as in the last case, $\frac{2}{b}\sqrt{(\frac{1}{4}b^2-\frac{1}{27}a^3)}$, or its equal $(\frac{27b^2-4a^3}{27b^2})^{\frac{1}{2}}=s$,

$$x = \pm \sqrt[3]{\frac{b}{2}}\left[\sqrt[3]{(1+s)} + \sqrt[3]{(1-s)}\right]$$

Whence, extracting the roots of the right hand member of this equation, there will arise

$$\sqrt[3]{(1+s)} = 1 + \frac{1}{3}s - \frac{2}{3.6}s^2 + \frac{2.5}{3.6.9}s^3 - \frac{2.5.8}{3.6.9.12}s^4 \&c.$$

$$\sqrt[3]{(1-s)} = 1 - \frac{1}{3}s - \frac{2}{3.6}s^2 - \frac{2.5}{3.6.9}s^3 - \frac{2.5.8}{3.6.9.12}s^4 \&c.$$

And, consequently, by adding the two series together, and taking the first term of the result as a multiplier, we shall have

$$x = \pm 2\sqrt[3]{\frac{b}{2}}\left\{1 - \frac{2}{3.6}s^2 - \frac{2.5.8}{3.6.9.12}s^4 - \frac{2.5.8.11.14}{3.6.9.12.15.18}s^6 \&c.\right\}$$

Or, by substituting $\left(\frac{27b^2 - 4a^3}{27b^2}\right)^{\frac{1}{2}}$, as given above, for its equal s,

$$x = \pm 2\sqrt[3]{\frac{b}{2}}\left[1 - \frac{2}{3.6}\left(\frac{27b^2 - 4a^3}{27b^2}\right) - \frac{2.5.8}{3.6.9.12}\left(\frac{27b^2 - 4a^3}{27b^2}\right)^2 - \frac{2.5.8.11.14}{3.6.9.12.15.18}\left(\frac{27b^2 - 4a^3}{27b^2}\right)^3 \&c.\right]$$

Which expression is likewise the same, for this case, as that given by the rule.

Also, if the equation be, as in the third, or irreducible case, of the form

$$x^3 - ax = \pm b,$$

where $\frac{1}{4}b^2$ is supposed to be less than $\frac{1}{27}a^3$, or $27b^2 = 4a^3$, we shall have, for the first of the three roots,

$$x = \pm 2\sqrt[3]{\frac{b}{2}}\left[1 + \frac{2}{3.6}\left(\frac{4a^3 - 27b^2}{27b^2}\right) - \frac{2.5.8}{3.6.9.12}\left(\frac{4a^3 - 27b^2}{27b^2}\right)^2 + \frac{2.5.8.11.14}{3.6.9.12.15.18}\left(\frac{4a^3 - 27b^2}{27b^2}\right)^3 \&c.\right]$$

Which expression is obtained from the last series, by barely changing the signs of the numerator and

denominator in each of its terms; which does not alter their value.

Hence, in order to determine the other two roots of the equation, let that above found, or its equivalent expression,

$$\sqrt[3]{[\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2-\tfrac{1}{27}a^3)}]}+\sqrt[3]{[\tfrac{1}{2}b-\sqrt{(\tfrac{1}{4}b^2-\tfrac{1}{27}a^3)}]}=\pm r.$$

Then, according to the formula that has been before given for these roots, in the former part of the present article, we shall have

$$x=\mp\frac{r}{2}\pm\frac{\sqrt{-3}}{2}\{\sqrt[3]{[\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2-\tfrac{1}{27}a^3)}]}-\sqrt[3]{[\tfrac{1}{2}b-\sqrt{(\tfrac{1}{4}b-\tfrac{1}{27}a^3)}]}\}$$

Or, putting $\frac{2}{b}\sqrt{(\frac{1}{4}b^2-\frac{1}{27}a^3)}=s$, and reducing the expression,

$$x=\mp\frac{r}{2}\pm\frac{(\frac{1}{3}b)^{\frac{1}{3}}\sqrt{-3}}{2}[\sqrt[3]{(1+s)}-\sqrt[3]{(1-s)}]$$

Whence, extracting the cube roots of the right hand member of this last equation, there will arise

$$\sqrt[3]{(1+s)}=1+\frac{1}{3}s-\frac{2}{3.6}s^2+\frac{2.5}{3.6.9}s^3-\frac{2.5.8}{3.6.9.12}s^4\&c.$$

$$\sqrt[3]{(1-s)}=1-\frac{1}{3}s-\frac{2}{3.6}s^3-\frac{2.5}{3.6.9}s^3-\frac{2.5.8}{3.6.9.12}s^4\&c.$$

And, consequently, by taking the latter of these series from the former, and making the first term of the remainder a multiplier, we shall have

$$x=\mp\frac{r}{2}\pm\frac{s(\frac{1}{3}b)^{\frac{1}{3}}\sqrt{-3}}{3}[1+\frac{2.5}{6.9}s^2+\frac{2.5.8.11}{6.9.12.15}s^4+\frac{2.5.8.11.14.17}{6.9.12.15.18.21}s^6\&c.].$$

But since $s=\frac{2}{b}\sqrt{(\frac{1}{4}b^2-\frac{1}{27}a^3)}=(\frac{27b^2-4a^3}{27b^2})^{\frac{1}{2}}$, $s^2=\frac{27b^2-4a^3}{27b^2}=-\frac{4a^3-27b^2}{27b^2}$, $s^4=(\frac{4a^3-27b^2}{27b^2})^2\&c.$, we have

also $\sqrt[3]{\frac{1}{2}b} \times \frac{3}{3}\sqrt{-3} = \sqrt[3]{\frac{1}{2}b} \times \frac{\sqrt{-3}}{3}\sqrt{\left(\frac{27b^2-4a^3}{27b^2}\right)} = \frac{1}{3}\sqrt[3]{\frac{1}{2}b} \times \sqrt{\frac{4a^3-27b^2}{9b^2}} = \frac{\sqrt{(4a^3-27b^2)}}{9\sqrt[3]{2b^2}}$.

Hence, if these values be substituted for their equals, in the last series, the result will give

$$x = \mp\frac{r}{2} \pm \frac{\sqrt{(4a^3-27b^2)}}{9\sqrt[3]{2b^2}}\left\{1 - \frac{2.5}{6.9}\left(\frac{4a^3-27b^2}{27b^2}\right) + \frac{2.5.8.11}{6.9.12.15}\left(\frac{4a^3-27b^2}{27b^2}\right)^2 - \frac{2.5.8.11.14.17}{6.9.12.15.18.21}\left(\frac{4a^3-27b^2}{27b^2}\right)^3 \&c.\right\}$$

Which expression, taking $-\frac{r}{2}$ when b is positive, and $+\frac{r}{2}$ when it is negative, is the same as that given in the practical rule, for the two remaining roots of the equation.

Again, for another solution of the same equation, we shall have, by transposing the terms of the common formula, as in the first case,

$$x = \sqrt[3]{\left\{\sqrt{\left(\frac{1}{4}b^2 - \frac{1}{27}a^3\right)} + \frac{1}{2}b\right\}} - \sqrt[3]{\left\{\sqrt{\left(\frac{1}{4}b^2 - \frac{1}{27}a^3\right)} - \frac{1}{2}b\right\}}.$$

Or, by putting, for the sake of simplicity, as before, $\sqrt{\left(\frac{1}{4}b^2 - \frac{1}{27}a^3\right)} = s$, and reducing the equation,

$$x = \sqrt[3]{s}\left\{\sqrt[3]{\left(1 + \frac{b}{2s}\right)} - \sqrt[3]{\left(1 - \frac{b}{2s}\right)}\right\}$$

Whence, extracting the roots of the right hand member, as in the former instances,

$$\sqrt[3]{\left(1 + \frac{b}{2s}\right)} = 1 + \frac{1}{3}\left(\frac{b}{2s}\right) - \frac{2}{3.6}\left(\frac{b}{2s}\right)^2 + \frac{2.5}{3.6.9}\left(\frac{b}{2s}\right)^3 - \frac{2.5.8}{3.6.9.12}\left(\frac{b}{2s}\right)^4 \&c.$$

$$\sqrt[3]{\left(1 - \frac{b}{2s}\right)} = 1 - \frac{1}{3}\left(\frac{b}{2s}\right) - \frac{2}{3.6}\left(\frac{b}{2s}\right)^2 - \frac{2.5}{3.6.9}\left(\frac{b}{2s}\right)^3 - \frac{2.5.8}{3.6.9.12}\left(\frac{b}{2s}\right)^4 \&c.$$

And, consequently, by subtracting the latter of these series from the former, and making the first term of the result a multiplier, we shall have

$$x=\frac{2bs^{\frac{1}{3}}}{6s}\{1+\frac{2.5}{6.9}(\frac{b}{2s})^2+\frac{2.5.8.11}{6.9.12.15}(\frac{b}{2s})^4+\frac{2.5.8.11.14.17}{6.9.12.15.18.21}(\frac{b}{2s})^6\&c.\}$$

But since $s=\sqrt{(\frac{1}{4}b^2+\frac{1}{27}a^3)}$, we have $(\frac{b}{2s})^2=\frac{27b^2}{27b^2-4a^3}=-\frac{27b^2}{4a^3-27b^2}$, $(\frac{b}{2s})^4=(\frac{27b^2}{4a^3-27b^2})^2\&c.$ & $\frac{2bs^{\frac{1}{3}}}{6s}=\frac{2b}{6s^{\frac{2}{3}}}=\frac{2b}{6\sqrt[3]{(\frac{1}{4}b^2-\frac{1}{27}a^3)}}=\frac{2b}{6\sqrt[3]{[-(\frac{1}{27}a^3-\frac{1}{4}b^2)]}}=-\frac{2b}{\sqrt[3]{[2(4a^3-27b^2)]}}$

Whence, if these values be substituted for their equals in the last series, there will arise

$$x=\mp\frac{2b}{\sqrt[3]{[2(4a^3-27b^2)]}}\{1-\frac{2.5}{6.9}(\frac{27b^2}{4a^3-27b^2})+\frac{2.5.8.11}{6.9.12.15}(\frac{27b^2}{4a^3-27b^2})^2-\frac{2.5.8.11.14.17}{6.9.12.15.18.21}(\frac{27b^2}{4a^3-27b^2})^3\&c.\}$$

Which is another expression for the first root of the proposed equation, agreeing with that given in the last case of the practical rule.

And if we put the root thus found, or its equivalent expression,

$$\sqrt[3]{\{\sqrt{(\frac{1}{4}b^2-\frac{1}{27}a^3)}+\frac{1}{2}b\}}-\sqrt[3]{\{\sqrt{(\frac{1}{4}b^2-\frac{1}{27}a^3)}-\frac{1}{2}b\}}=\pm r,$$

we shall have, according to the formula before given for the other two roots,

$$x=\mp\frac{r}{2}\pm\frac{\sqrt{-3}}{2}\{\sqrt[3]{[\sqrt{(\frac{1}{4}b^2-\frac{1}{27}a^3)}+\frac{1}{2}b]}+\sqrt[3]{[\sqrt{(\frac{1}{4}b^2-\frac{1}{27}a^3)}-\frac{1}{2}b]}\}$$

Or taking, as before, $\sqrt{(\frac{1}{4}b^2-\frac{1}{27}a^3)}=s$, and simplifying the result,

$$x = \mp\frac{r}{2} \pm \frac{s^{\frac{1}{2}}\sqrt{-3}}{2}\{\sqrt[3]{(1+\frac{b}{2s})} + \sqrt[3]{(1-\frac{b}{2s})}\}$$

Whence, by extracting the roots of the right hand side of this equation, there will arise

$$\sqrt[3]{(1+\frac{b}{2s})} = 1 + \frac{1}{3}(\frac{b}{2s}) - \frac{2}{3.6}(\frac{b}{2s})^2 + \frac{2.5}{3.6.9}(\frac{b}{2s})^3 - \frac{2.5.8}{3.6.9.12}(\frac{b}{2s})^4 \&c.$$

$$\sqrt[3]{(1-\frac{b}{2s})} = 1 - \frac{1}{3}(\frac{b}{2s}) - \frac{2}{3.6}(\frac{b}{2s})^2 - \frac{2.5}{3.6.9}(\frac{b}{2s})^3 - \frac{2.5.8}{3.6.9.12}(\frac{b}{2s})^4 \&c.$$

And, consequently, if the latter of these series be added to the former, we shall have, by making the first term of the result a multiplier,

$$x = \mp\frac{r}{2} \pm s^{\frac{1}{2}}\sqrt{-3}\{1 - \frac{2}{3.6}(\frac{b}{2s})^2 - \frac{2.5.8}{3.6.9.12}(\frac{b}{2s})^4 - \frac{2.5.8.11.14}{3.6.9.12.15.18}(\frac{b}{2s})^6 \&c.\}$$

But since $s = \sqrt{(\frac{1}{4}b^2 - \frac{1}{27}a^3)} = (\frac{27b^2 - 4a^3}{4.27})^{\frac{1}{2}}$, we have also $(\frac{b}{2s})^2 = \frac{27b^2}{27b^2 - 4a^3} = -\frac{27b^2}{4a^3 - 27b} \&c.$ and $s^{\frac{1}{2}}\sqrt{-3} = \sqrt[6]{\frac{4a^3 - 27b^2}{4}}$.

Hence, if these values be substituted for their equals in the above series, the result will give

$$x = \mp\frac{r}{2} \pm \sqrt[6]{\frac{4a^3 - 27b^2}{4}}\{1 + \frac{2}{3.6}(\frac{27b^2}{4a^3 - 27b^2}) - \frac{2.5.8}{3.6.9.12}(\frac{27b^2}{4a^3 - 27b^2})^2 + \frac{2.5.8.11.14}{3.6.9.12.15.18}(\frac{27b^2}{4a^3 - 27b^2})^3 \&c.\}$$

Which expression, taking $-\frac{r}{2}$, or $+\frac{r}{2}$, as before, is the same as that given in the last case of the practical rule, for the two remaining roots of the equation.

BIQUADRATIC EQUATIONS.

(R) VARIOUS methods having been proposed, by different authors, for the resolution of biquadratic equations, I shall here give, in as perspicuous a manner as possible, the investigations of such of them as appear to be most deserving of notice; leaving it to the judgment of the reader to decide upon their respective merits.

In the first place, then, it may be observed, that the earliest solution of equations of this kind, that has come to our knowledge, is that of Louis Ferrari, given by Cardan, in the 39th chapter of his Algebra, published at Milan, 1545; which is as follows:

Let there be taken the common numeral equation

$$x^4 + 6x^2 + 36 = 60x,$$

which is one among several of those proposed in the above mentioned work.

Then, if $6x^2$ be added to each of its sides, the left hand member will be a complete square; the result being

$$x^4 + 12x^2 + 36, \text{ or } (x^2 + 6)^2 = 6x^2 + 60x.$$

And, by taking any unknown quantity y, and adding $2y(x^2 + 6) + y^2$ to each side of this last equation, it will become

$$(x^2 + 6)^2 + 2y(x^2 + 6) + y^2 = 6x^2 + 60x + 2y(x^2 + 6) + y^2.$$

Where, since the left hand member is still a square, we shall have, by simplifying the equation,

$$(x^2+6+y)^2=$$
$$(6+2y)x^2+60x+12y+y^2. \quad . \quad . \quad . \quad (1)$$

Hence it only remains to make the right hand member also a square; which is done by putting the 1st term $\times$ 3d $= \frac{1}{4}$sq. 2d, or

$$(6+2y)x^2 \times (12y+y^2)=\tfrac{1}{4}(60x)^2.$$

And, consequently, by actually multiplying the two compound terms of this expression together, and squaring the 3d, we shall have, after expunging the common factor $2x^2$,

$$y^3+15y^2+36y=450.$$

The root of which last equation, as found by the rule before given for cubics, will be

$$y=$$
$$\sqrt[3]{\{287\tfrac{1}{2}+\sqrt{(80449\tfrac{1}{4})}\}}+\sqrt[3]{\{287\tfrac{1}{2}-\sqrt{(80449\tfrac{1}{4})}\}}-5.$$

And, therefore, since equation 1, (which by the above process has each of its sides a square), can be put under the form

$$(x^2+6+y)^2=\{\sqrt{(6+2y)}\times x+\sqrt{(12y+y^2)}\}^2,$$

we shall have, by extracting the square root of each of its two members,

$$x^2+6+y=\pm\{\sqrt{(6+2y)}\}x+\sqrt{(12y+y^2)}$$

for the case when $60x$ in the given equation is positive; and

$$x^2+6+y=\pm\{\sqrt{(6+2y)}\}x-\sqrt{(12y+y^2)}$$

for the case when the same quantity is negative.

From which two quadratics, and the above value of y, the four roots of the proposed biquadratic can be readily found (*k*).

(*k*) This method of resolving Biquadratic Equations, which was afterwards copiously explained and illustrated by Bombelli,

The second method, is that given by Descartes, in the third book of his Geometry, first published in 1637; which is founded upon the following principles.

Let the equation, whose roots are required to be found, be of the form

$$x^4 + ax^2 + bx + c = 0,$$

in which the second term is wanting; and assume, as a first step in the investigation,

$$(x^2 + px + q) \times (x^2 + rx + s) = x^4 + ax^2 + bx + c.$$

Or, which is the same thing, let the proposed biquadratic be considered as produced by the multiplication of the two quadratics

$$x^2 + px + q = 0, \text{ and } x^2 + rx + s = 0.$$

Then, since these equations, as well as the given one, are each $= 0$, there will arise, by taking their product,

$$x^4 + (p + r)x^3 + (s + q + pr)x^2 + (ps + qr)x + qs =$$
$$x^4 + ax^2 + bx + c.$$

And, consequently, by equating the homologous terms of this last equation, we shall have the four following equations,

$$p + r = 0;\ s + q + pr = a;\ ps + qr = b;\ qs = c.$$

Or, if $-p$, which is the value of r in the first of these, be substituted for r in the second and third, they will become

$$s + q = a + p^2;\ s - q = \frac{b}{p};\ qs = c.$$

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in his work before mentioned, continued in use among mathematicians for near a century after its first invention by Ferrari; and is, perhaps, in its general principles, inferior to none that has yet been discovered.

Whence, subtracting the square of the second of these from that of the first, and then changing the sides of the equation, we shall have

$$a^2 + 2ap^2 + p^4 - \frac{b^2}{p^2} = 4qs, \text{ or } 4c.$$

And, therefore, by multiplying by p^2, and placing the terms according to the order of their powers, the result will give

$$p^6 + 2ap^4 + (a^2 - 4c)p^2 = b^2.$$

From which last equation, if there be put $p^2 = z$, we shall have

$$z^3 + 2az^2 + (a^2 - 4c)z = b^2;$$

where z, and consequently p, may be found by the rule before given for cubic equations.

Hence, also, since $s + q = a + p^2$, and $s - q = \frac{b}{p}$, there will arise, by addition and subtraction,

$$s = \frac{1}{2}a + \frac{1}{2}p^2 + \frac{b}{2p};\ q = \frac{1}{2}a + \frac{1}{2}p^2 - \frac{b}{2p};$$

where p being known, s and q are likewise known.

And, consequently, by extracting the roots of the two assumed quadratics $x^2 + px + q = 0$, and $x^2 + rx + s = 0$, or its equal $x^2 - px + s = 0$, we shall have

$$x = -\tfrac{1}{2}p \pm \sqrt{(\tfrac{1}{4}p^2 - q)};\ x = \tfrac{1}{2}p \pm \sqrt{(\tfrac{1}{4}p^2 - s)};$$

which expressions, when taken in + and −, give the four roots of the proposed biquadratic, as was required (*l*).

(*l*) Since, in the above investigation, $p + r = 0$, and $qs = c$, it is evident that the reducing equation in r will be the same as in p, and that in q the same as in s; but the latter is of a form which

It may here be observed, however, that this solution, as well as the four following ones, can only be applied to those cases, in which two roots of the biquadratic are possible and two impossible.

For let us suppose, as has been done by Wood, p. 156 of his Algebra, that the four roots of the biquadratic are a, b, c, and $-(a+b+c)$.

Then, since p, the coefficient of the second term of one of the reducing quadratics, is the sum of two of its roots, all the different values of which this sum is susceptible are

$a+b$, $a+c$, $b+c$, $-(a+b)$, $-(a+c)$, and $-(b+c)$.

And, consequently, all the different values that p^2, or z, can admit of, are

$$(a+b)^2,\ (a+c)^2,\ \text{and}\ (b+c)^2.$$

Whence, as these are all possible, the cubic falls

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was considered by Waring (p. 135, Meditationes Algebraicæ) and other writers, as irresolvable, being

$$s^6 - as^5 - cs^4 + (2ac - b^2)s^3 - c^2s^2 - ac^2s + c^3 = 0.$$

But, by dividing it by s^3, and arranging the terms according to the powers of s and their reciprocals, as was first shown by Woodhouse, it will become of the form

$$s^3 + \frac{c^3}{s^3} - a\left(s^2 + \frac{c^2}{s^2}\right) - c\left(s + \frac{c}{s}\right) + 2ac - b^2 = 0;$$

where, putting $s + \frac{c}{s} = v$, we shall have $s^2 + \frac{c^2}{s^2} = v^2 - 2c$, and $s^3 + \frac{c^3}{s^3} = v^3 - 3cv$; and, consequently, by substitution,

$$v^3 - av^2 - 4cv + 4ac - b^2 = 0.$$

Whence, since $s + \frac{c}{s} = v$, or $s^2 - vs + c = 0$, if the value of v in the former of these equations be substituted for v in the latter, the value of s may be readily determined.

under the irreducible case, and therefore cannot be resolved by the common formula, except in some particular cases.

Again, suppose the roots of the biquadratic to be

$$a+b\sqrt{-1},\ a-b\sqrt{-1},\ -a+c\sqrt{-1},\ \text{and}$$
$$-a-c\sqrt{-1}.$$

Then, taking the sums of them as before, the different values of p^2, or z, will be found to be

$$(2a)^2,\ -(b+c)^2,\ \text{and}\ -(b-c)^2;$$

which are all possible, as in the preceding case.

But if the roots of the given biquadratic be

$$a+b\sqrt{-1},\ a-b\sqrt{-1},\ -a+c,\ \text{and}\ -a-c,$$

the values of p^2, or z, will, in that case, be

$$(2a)^2,\ (c+b\sqrt{-1})^2,\ \text{and}\ (c-b\sqrt{-1})^2;$$

two of which being impossible, the cubic may be resolved by Cardan's rule.

And the same may be shown to take place, in nearly a similar manner, in all the following methods.

The third method is that given by Simpson, p. 150 of his Algebra, second edition, which consists in supposing the given biquadratic to be formed by taking the difference of two complete squares, being the same in principle as that of Ferrari.

Thus, let the proposed equation be of the form

$$x^4+ax^3+bx^2+cx+d=0,$$

having all its terms complete; and assume

$$(x^2+\tfrac{1}{2}ax+p)^2-(qx+r)^2=x^4+ax^3+bx^2+cx+d.$$

Then, if $x^2+\frac{1}{2}ax+p$ and $qx+r$ be actually involved, we shall have

$$\left.\begin{array}{l|l|l} x^4+ax^3+2p & x^2+ap & x+p^2 \\ \quad\quad\quad +\frac{1}{4}a^2 & \quad -2qr & \quad -r^2 \\ \quad\quad\quad\quad -q^2 & & \end{array}\right\} = x^4+ax^3+bx^2+cx+d.$$

And, consequently, by equating the homologous terms, there will arise

$$\begin{array}{l|c|l} 1.\ 2p+\frac{1}{4}a^2-q^2=b & & -2p+\frac{1}{4}a^2-b=q^2 \\ 2.\ ap-2qr \quad\quad =c & \text{or} & ap-c \quad\quad =2qr \\ 3.\ p^2-r^2 \quad\quad\quad =d & & p^2-d \quad\quad =r^2 \end{array}$$

Where, since the product of the first and last of these equations is evidently equal to $\frac{1}{4}$ of the square of the second, we shall have

$$2p^3+(\tfrac{1}{4}a^2-b)p^2-2dp-d(\tfrac{1}{4}a^2-b)=$$
$$\tfrac{1}{4}(a^2p^2-2acp+c^2).$$

Or, by bringing the unknown quantities to the left hand side, and the known quantities to the right, and then dividing by 2,

$$p^3-\frac{b}{2}p^2+\frac{1}{4}(ac-4d)p=\frac{1}{8}(c^2+a^2d)-\frac{1}{2}bd.$$

From which last equation p can be determined, as in the former method, by the rule before given for cubics.

And since, from the preceding equations, it appears that

$$q=\sqrt{(2p+\frac{1}{4}a^2-b)} \text{ and } r=\frac{ap-c}{2q}, \text{ or } \sqrt{(p^2-d)},$$

it is evident that the several values of x can be readily obtained from the quantities thus found.

For, because $x^4+ax^3+bx^2+cx+d$, or its equal $(x^2+\frac{1}{2}ax+p)^2-(qx+r)^2=0$, it is plain that

$$(x^2+\tfrac{1}{2}ax+p)^2=(qx+r)^2.$$

And, therefore, by extracting the roots of each side of this equation, there will arise

$$x^2+\tfrac{1}{2}ax+p=qx+r;\ \text{or}\ x^2+(\tfrac{1}{2}a-q)x=r-p.$$

Whence, by substituting the above values of p, q, and r, and transposing the terms, we shall have

$$x^2+\{\tfrac{1}{2}a\mp\sqrt{(2p+\tfrac{1}{4}a^2-b)}\}x+p\mp\sqrt{(p^2-d)}=0,$$

for the case where $ap-c$ is positive; and

$$x^2+\{\tfrac{1}{2}a\mp\sqrt{(2p+\tfrac{1}{4}a^2-b)}\}x+p\pm\sqrt{(p^2-d)}=0,$$

for the case where $ap-c$ is negative.

Which two quadratics give the four roots of the proposed equation.

And by putting $p=z+\frac{b}{6}$, in order to destroy the second term of the reducing cubic, the several steps of the investigation may be readily made to agree with the expressions given in the practical rule, Part 1 (*m*).

The fourth method is that given by Waring, in chap. iv of his Miscellanea Analytica, published at Cambridge in 1762; which, being also a generalization of the one before ascribed to Ferrari, may be investigated in a similar manner, as follows:

(*m*) If each of the three values of p, in the reducing cubic above found, render the expression $\sqrt{(2p+\tfrac{1}{4}a^2-b)}=0$, this method of resolution will evidently fail, as the coefficient q, in that case, becomes indeterminate.

But this can only happen when the equation is of the form

$$x^4+ax^3+\frac{3}{8}a^2x^2+\frac{1}{16}a^3x+\frac{1}{256}a^4=0;$$

in which case it has each of its roots $=-\tfrac{1}{4}a$.

The same restriction also takes place in the rule given by Waring, next following, as also in the extension of that of Descartes, to the form $x^4+ax^3+bx^2+cx+d=0$, as given by Maclaurin, p. 229 of his Algebra.

Let the proposed biquadratic be of the form

$$x^4 + ax^3 = bx^2 + cx + d.$$

Then, since by squaring the assumed expression $x^2 + \frac{1}{2}ax + p$, we shall have

$$(x^2 + \tfrac{1}{2}ax + p)^2 = x^4 + ax^3 + (\tfrac{1}{4}a^2 + 2p)x^2 + apx + p^2.$$

If $(\frac{1}{4}a^2 + 2p)x^2 + apx + p^2$ be added to each side of the original equation, it will become

$$(x^2 + \tfrac{1}{2}ax + p)^2 =$$

$$(\tfrac{1}{4}a^2 + 2p + b)x^2 + (ap + c)x + p^2 + d, \quad . \quad . \quad . \quad (1)$$

where, since the first member is a square, it only remains to make the second a square; which is done by taking

$$4(p^2 + d) \times (\tfrac{1}{4}a^2 + 2p + b) = (ap + c)^2.$$

Whence, multiplying the two compound terms of this expression together, and squaring the third, we shall have

$$8p^3 + (a^2 + 4b)p^2 + 8dp + a^2d + 4bd = a^2p^2 + 2acp + c^2.$$

Or, bringing all the unknown quantities to the left hand side, and the known quantities to the right, and then dividing by 8,

$$p^3 + \frac{1}{2}bp^2 + \frac{1}{4}(4d - ac)p = \frac{1}{8}c^2 - \frac{d}{8}(a^2 + 4b).$$

From which last equation the value of p can be determined, as in the former methods.

And since equation 1, from its right hand member having been made a square, can be put under the form

$$(x^2 + \tfrac{1}{2}ax + p)^2 = \{\sqrt{(\tfrac{1}{4}a^2 + 2p + b)} \times x + \sqrt{(p^2 + d)}\}^2,$$

we shall have, by extracting the roots of its two sides,

$$x^2 + \tfrac{1}{2}ax + p = \pm\{\sqrt{(\tfrac{1}{4}a^2 + 2p + b)}\}x + \sqrt{(p^2 + d)}, \text{ or}$$

$$x^2 + \{\tfrac{1}{2}a \mp \sqrt{(\tfrac{1}{4}a^2 + 2p + b)}\}x = -p + \sqrt{(p^2 + d)},$$

for the case where the second term $ap+c$, of equation (1) is positive; and

$$x^2+\tfrac{1}{2}ax+p=\pm\{\sqrt{(\tfrac{1}{4}a^2+2p+b)}\}x-\sqrt{(p^2+d)}, \text{ or}$$
$$x^2+\{\tfrac{1}{2}a\mp\sqrt{(\tfrac{1}{4}a^2+2p+b)}\}x=-p-\sqrt{(p^2+d)},$$

for the case where the same quantity is negative: which two quadratics will, therefore, give the four roots of the proposed biquadratic, as was required.

The fifth and last method, which it will be here necessary to explain, is that of Euler, first given by him in vol. vi of the ancient Commentaries of Petersburgh, 1738; and, afterwards, more at large, in his Treatise of Algebra, which was translated from the German into French in 1774, and since into English, in 2 vols. 8vo.

This method, which differs considerably from either of the former, consists in supposing the root of the given equation

$$x^4+ax^2+bx+c=0,$$

to be of the trinomial surd form

$$x=\sqrt{p}+\sqrt{q}+\sqrt{r};$$

where p, q, r, denote the roots of the cubic equation

$$y^3+fy^2+gy=h,$$

of which the coefficients f, g, and the absolute term h, are the unknown quantities that are to be determined.

Then, agreeably to the theory of equations before given, we shall have

$$p+q+r=-f;\ pq+pr+qr=g;\ pqr=h.$$

And, by squaring each side of the formula, expressing the value of x,

$$x^2=p+q+r+2\sqrt{pq}+2\sqrt{pr}+2\sqrt{qr}.$$

Or, substituting f for its equal $-(p+q+r)$, and bringing the term, so obtained, to the other side of the equation

$$x^2+f=2\sqrt{pq}+2\sqrt{pr}+2\sqrt{qr}.$$

Also, by again squaring each side of this last expression, we shall have

$$x^4+2fx^2+f^2=$$
$$4pq+4pr+4qr+8\sqrt{p^2qr}+8\sqrt{q^2pr}+8\sqrt{r^2pq}.$$

Or, substituting $4g$ for its equal $4pq+4pr+4qr$, and bringing the term to the other side, as before,

$$x^4+2fx^2+f^2-4g=8\sqrt{pqr}(\sqrt{p}+\sqrt{q}+\sqrt{r}).$$

But since, from what has been above laid down, we have

$$\sqrt{p}+\sqrt{q}+\sqrt{r}=x, \text{ and } \sqrt{pqr}=\sqrt{h},$$

if these be put for their equals in the last equation, it will become

$$x^4+2fx^2-8h^{\frac{1}{2}}x+f^2-4g=0.$$

Whence, comparing these coefficients with those of the given equation, there will arise

$$2f=a;\ -8\sqrt{h}=b;\ f^2-4g=c.$$

Or

$$f=\frac{a}{2};\ h=\frac{b^2}{64};\ g=\frac{a^2}{16}-\frac{c}{4}.$$

And, consequently, by substituting these values in the assumed cubic $y^3+fy^2+gy=h$, we shall have

$$y^3+\frac{1}{2}ay^2+\frac{1}{16}(a^2-4c)y=\frac{b^2}{64}.$$

The three roots of which last equation, when substituted for p, q, and r, in the first formula, will give, by taking each term of the expression both in $+$ and $-$, all the four values of x.

Or, in order to render this result more commodious in practice, by freeing it from fractions, let

$$y = \tfrac{1}{4}z.$$

Then, by substitution and reduction, we shall have the corresponding equation

$$z^3 + 2az^2 + (a^2 - 4c)z = b^2,$$

the three roots of which are each evidently equal to $\frac{1}{4}$th of those of the former.

Hence, using this equation instead of the other, and denoting its roots by r', r'', r''', the assumed formula, taking each of its terms in + and −, as before, will give the following values of x.

1. *When b is positive.*

$$x = \tfrac{1}{2}(-\sqrt{r'} - \sqrt{r''} - \sqrt{r'''})$$
$$x = \tfrac{1}{2}(-\sqrt{r'} + \sqrt{r''} + \sqrt{r'''})$$
$$x = \tfrac{1}{2}(+\sqrt{r'} - \sqrt{r''} + \sqrt{r'''})$$
$$x = \tfrac{1}{2}(+\sqrt{r'} + \sqrt{r''} - \sqrt{r'''})$$

2. *When b is negative.*

$$x = \tfrac{1}{2}(+\sqrt{r'} + \sqrt{r''} + \sqrt{r'''})$$
$$x = \tfrac{1}{2}(+\sqrt{r'} - \sqrt{r''} - \sqrt{r'''})$$
$$x = \tfrac{1}{2}(-\sqrt{r'} + \sqrt{r''} - \sqrt{r'''})$$
$$x = \tfrac{1}{2}(-\sqrt{r'} - \sqrt{r''} + \sqrt{r'''})$$

Which expressions are the same as those before given, in the practical rule, Part I (*n*).

(*n*) If we were to take all the possible changes of the signs, in this case, which the terms of the assumed formula admit of, it would appear that x should have eight different values; but it is to be observed, that, according to the first part of the above

RESOLUTION OF NUMERAL EQUATIONS OF THE HIGHER ORDERS.

(s) The foregoing articles will be found to contain most of the useful formulæ that have hitherto been devised for resolving equations of the first four degrees; but for those of the fifth and other higher orders, no general rule, as before observed, has yet been discovered, by which their roots can be determined; though it is well known, that every equation, whatever may be the number of its dimensions, must have, at least, one root, either real or imaginary.

Thus, for example, the following equation of the fifth power,

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0,$$

has been shown, in Art. (N) 14, to have, at least, one real root, which must necessarily depend upon the

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investigation, the product $\sqrt{p} \times \sqrt{q} \times \sqrt{r} = \sqrt{h}$, or $\frac{1}{8}b$; and, consequently, when b is positive, either all the three radicals must be taken in $+$, or two in $-$ and one in $+$; and when b is negative, they must either be all $-$, or two $+$ and one $-$; which considerations reduce the number of roots to four.

Mr. Lea, in a very ingenious small tract, entitled *A Treatise on the Resolution of the higher Equations in Algebra*, of some particulars of which I have availed myself in the present article, has shown how all the methods here given for solving cubics and biquadratics, as well as those proposed by Tschirnhaus, Bezout, Euler, and Waring, for the resolution of equations in general, may all be derived from one principle.

He has here likewise been the first to remark, that the method of surd divisors, so largely treated on by Maclaurin, Simpson, and other writers, is, with respect to biquadratic equations, entirely useless.

values of its coefficients a, b, c, &c.; because x will evidently vary as these vary; but, notwithstanding this certainty, we are still unable to exhibit the value of x, in any function of the quantities a, b, c, d, e, however they may be expressed or combined.

Hence, in cases of this kind, where no general formula can be obtained, we must content ourselves with finding, either exactly or by approximation, the roots of such equations only as have numeral coefficients, or of those that can be reduced to that state; which is, indeed, all that is required for the final solution of any analytical problem (*o*).

With this view, therefore, it may be observed, that if any of the roots of an equation, of the common form, be whole numbers, they may be determined by substituting 1, 2, 3, 4, &c. successively, both in *plus* and in *minus*, for the unknown quantity, till a result is obtained equal to that in the question; when those that are found to succeed, will be the roots required.

But as the number of substitutions, which it would be necessary to make in this way, will be often very great, the trouble arising from so many operations

(*o*) Lagrange has observed, in his treatise *De la resolution des équation numérique*, p. 7, that, if we could by any means arrive at a general solution of equations of the fifth and other higher dimensions, we should only obtain certain formulæ, which, though highly valuable in themselves, would be but of little use in the effective resolution of numeral equations of these orders, as recourse must still be had, for the sake of a more easy computation, to some of the methods of approximation usually given.

may, in most cases, be considerably abridged, by first finding *the limits* of the roots of the proposed equation, according to Art. (N) 12, and then substituting only the whole numbers that lie between them; among which all the commensurate roots must necessarily fall.

Or, since the last term of any equation is always equal to the continued product of all its roots, Art. (N) 3, these trials may be frequently still farther diminished, by finding all the divisors of that term, and then substituting such of them, for the unknown quantity, as are comprised within the abovementioned limits; when those that give the proper result will be the rational roots sought: but if none of them are found to succeed, it may be concluded that the equation cannot be resolved by this method; the roots, in that case, being either irrational or imaginary.

Thus, let the equation, which is proposed to be resolved, be

$$x^3 - 4x^2 - 7x + 10 = 0;$$

Then all the divisors of the last term, in this case, will be

$$+1,\ -1,\ +2,\ -2,\ +5,\ -5,\ +10,\ -10.$$

And, consequently, by substituting as many of these numbers for x, as are necessary for obtaining the three roots of the equation, we shall have

$+1$	$+1-4-7+10=0$
-1	$-1-4+7+10=12$
$+2$	$+8-16-14+10=-12$
-2	$-8-16+14+10=0$
$+5$	$+125-100-35+10=0$

Whence, as $+1$, -2, and $+5$, give results $=0$,

which is that of the proposed equation, they must necessarily be the roots sought.

In this case, also, if the divisors of the last term be too numerous, they may be diminished by transforming the given equation into another, that shall have its last term less than that of the former; which may be done by increasing or diminishing the roots by 1, or some other known quantity, as in Art. (N) 8.

Thus, let there be taken, as an example of this kind, the equation

$$x^4 - 4x^3 - 8x + 32 = 0.$$

Then, in order to change it into another, the last term of which shall have a less number of divisors, substitute $y + 1$ for x; and the transformed equation, in this case, will be

$$y^4 - 6y^2 - 16y + 21 = 0.$$

Where the several divisors of the last term will now be

$$1, -1, 3, -3, 7, -7, \text{ and } 21, -21.$$

Which being successively substituted for y, we shall have

$+1$	$1 - 6 - 16 + 21 = 0$
-1	$1 - 6 + 16 + 21 = 32$
$+3$	$81 - 54 - 48 + 21 = 0$

And, consequently, from this part of the process, it appears that 1 and 3 are two of the roots of the equation.

But as 7 and -7, 21 and -21, will be found, upon trial, to give results greater or less than 0, the other two roots must be either irrational or imaginary.

In effect, their sum being $= -(1 + 3)$, and their

product $= \frac{21}{3}$, or 7, they must necessarily be both imaginary; and, if determined according to the above conditions, will be found to be $-2+\sqrt{-3}$, and $-2-\sqrt{-3}$.

Whence, if each of these roots be increased by 1, we shall have 2, 4, $-1+\sqrt{-3}$, and $-1-\sqrt{-3}$, for the four roots of the given equation.

Also, if the proposed equation be a literal one, which has all its terms homogeneous, or of the form

$$x^m + pax^{m-1} + qa^2x^{m-2} + ra^3x^{m-3} + \ldots + va^m = 0,$$

it may be resolved in the same manner, by finding all the divisors of the numeral part of the last term, as before, and then multiplying such of them as are found to succeed, by the known quantity, for the root.

Thus, if there be taken the following literal equation of this kind,

$$x^3 + 3ax^2 - 4a^2x - 12a^3 = 0,$$

we shall have, for the divisors of the numeral part of the last term,

$$1,\ -1,\ 2,\ -2,\ 3,\ -3,\ \&c.$$

And if these be substituted successively for x, without regarding the known quantity a, there will arise

1	$1+3-4-12=-12$
-1	$-1+3+4-12=-6$
2	$8+12-8-12=0$
-2	$-8+12+8-12=0$
3	$27+27-12-12=30$
-3	$-27+27+12-12=0$

Whence, the numbers that succeed in giving

results $=0$, being 2, -2, and -3, the roots of the given equation will be $2a$, $-2a$, and $-3a$.

The integral roots of any numeral equation, of the kind above mentioned, may also be found, in a different manner, by what is usually called the *method of divisors*; which is founded upon the following principles.

1. Let any equation, similar to those before given, be represented under the form

$$(x+a)\mathrm{Q}=0,$$

where the binomial $x+a$ denotes one of the factors of which the equation is composed, and Q the product of the rest.

Then, if three or more terms of the arithmetical series 2, 1, 0, -1, -2, be successively substituted for x, the divisors of the results, thus obtained, will be

$$a+2,\ a+1,\ a,\ a-1, \text{ and } a-2.$$

And as these are, also, in arithmetical progression, it is plain that the roots of the given equation, when integral, will be some of the numbers in such a series.

Whence, if a progression of this kind, whose common difference is 1, can be found among the divisors of the results above mentioned, by taking one number out of each of the lines, that term of it which answers to the substitution of 0 for x, taken in $+$ or -1, according as the series is increasing or decreasing, will be generally a root of the equation.

Thus, if the equation which is proposed to be resolved, be

$$x^3-x^2-10x+6=0,$$

the operation, according to the method above described, will stand as follows:

Nos.	*Results.*	*Divisors.*	*Prog.*
2	−10	1 . 2 . 5 . 10	5
1	−4	1 . 2 . 4 . —	4
0	+6	1 . 2 . 3 . 6	3
−1	+14	1 . 2 . 7 . 14	2
−2	+14	1 . 2 . 7 . 14	1

Where the only arithmetical progression that can be formed from the divisors of the several results in the second column, by taking one term out of each line, is 5, 4, 3, 2, 1; and therefore, as the series is decreasing, −3, which is the number that stands opposite to 0, will be found, upon trial, to be a root of the equation.

In like manner, if the equation that is to be resolved, were

$$x^4 + x^3 - 29x^2 - 9x + 180 = 0,$$

the work, agreeably to the same mode of proceeding, would stand thus:

Nos.	*Results.*	*Divisors.*	*Progressions.*			
2	70	1 . 2 . 5 . 7 . 10 . 14 &c.	1	2	5	7
1	144	1 . 2 . 3 . 4 . 6 . 8 &c.	2	3	4	6
0	180	1 . 2 . 3 . 4 . 5 . 6 &c.	3	4	3	5
−1	160	1 . 2 . 4 . 5 . 8 . 10 &c.	4	5	2	4
−2	90	1 . 2 . 3 . 5 . 6 . 9 &c.	5	6	1	3

Where there are no less than four progressions, whose terms differ by unity, that can be formed out of the results in the second column; and as the two former are ascending ones, and the two latter descending, +3, +4, −3, and −5, will be found to be the roots of the equation.

2. If the highest power of the unknown quantity has any coefficient prefixed to it, let the equation be assumed of the form

$$(nx+a)a=0,$$

and substitute 2, 1, 0, -1, -2, successively for x, as before.

Then, if $-\frac{a}{n}$ be a root of the equation, the divisors of the several results, arising from this substitution, will be the terms of the arithmetical series

$$2n+a,\ n+a,\ a,\ -n+a, \text{ and } -2n+a;$$

where the common difference n must be a divisor of the first term of the equation, or otherwise the division would not succeed.

Hence, in this case, the progressions must be so taken out of the divisors, that their terms shall differ from each other by some aliquot part of that coefficient.

And, therefore, if the terms of these series, which stand opposite to 0, be divided by the common difference, the quotients thus arising, taken in + or −, according as the progression is increasing or decreasing, will, in most cases, be the roots of the equation (*p*).

(*p*) The reason why the numbers, thus found, do not always succeed, is because the first series 2, 1, 0, -1, -2, is not continued far enough to show whether the corresponding progression may not break off, after a certain number of terms; which it never can do when it contains a real rational root. Thus, in the example given in the following pages, if the operation had been continued only one term farther, the first of these progressions 2, 3, 4, 5, 6, would have been found to fail; whereas, on the contrary,

Thus, let there be taken, as an example of this kind, the equation

$$2x^3 - 3x^2 + 16x - 24 = 0.$$

Then, substituting 2, 1, 0, -1, and -2, successively for x, as in the former cases, we shall have,

Nos.	*Results.*	*Divisors.*	*Progressions.*	
2	12	1 . 2 . 3 . 4 . 6 . 12	2	-1
1	-9	1 . 3 . 9	3	$+1$
0	-24	1 . 2 . 3 . 4 . 6 . 8 &c.	4	$+3$
-1	-45	1 . 3 . 5 . 9 . 15 . 45	5	$+5$
-2	-84	1 . 2 . 3 . 4 . 6 . 7 &c.	6	$+7$

Where the progressions being both ascending, the numbers to be tried are $\frac{4}{1}$, or 4, and $\frac{3}{2}$; but the latter of these only is a root, as the other will be found, by substituting it for x, to give 120 for the result, instead of 0, as it ought.

RESOLUTION OF EQUATIONS BY APPROXIMATION.

(T) The methods laid down in the foregoing Article, will be found sufficient for determining the integral, or commensurate roots of equations of all orders; but for those which have their roots

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the latter, -1, 1, 3, 5, 7, will hold, however far the progression 2, 1, 0, -1, -2 may be carried.

It may here, also, be farther remarked, that if an equation, of four or more dimensions, does not admit of a binomial factor of the form $nx + r$, it may have a trinomial one of the form $mx^2 + nx + r$; and so on; but this mode of treating the subject, though not devoid of ingenuity, is of too little practical importance to require any particular notice in this place.

irrational, or incommensurable, recourse must be had to a different process; as they can then only be obtained by what is usually called the method of approximation; which, as far as regards the practical rule that has been given under this head, in the former part of the work, may be explained as follows.

Let there be taken the following general equation

$$x^m + px^{m-1} + qx^{m-2} + \ldots + sx^2 + tx + u = 0.$$

Then, supposing a to be a near value of x, found by trial, and z the remaining part of the root, we shall have (q)

$$x = a + z.$$

And, consequently, by substituting this value for x in the given equation, there will arise

$$(a+z)^m + p(a+z)^{m-1} + \ldots + s(a+z)^2 + t(a+z) + u = 0.$$

Which last expression, by involving its terms, and taking the result in an inverse order, may be transformed into the equation

$$\text{P} + \text{Q}z + \text{R}z^2 + \text{S}z^3 + \text{T}z^4 + \ldots + z^m = 0.$$

Where P, Q, R, &c. are polynomials, composed of

(q) The first approximate value of the unknown quantity, here mentioned, may, in general, be readily found, by taking two such numbers in the series 0, 1, 2, 3, &c. as when substituted for x, will give results with contrary signs; in which case, the root sought, when positive, will lie between these numbers, being the nearest to that which gives a result differing the least from the one in the question.

And if the successive substitutions of 0, 1, 2, 3, &c. for x, give only positive results, as they will always do, when the given equation has two, or some other even number of roots, that differ from each other by less than unity Art. (N) 19, the root sought will

certain functions of the known terms a, m, p, q, r, &c. which are derived from each other, according to a regular law.

Thus, by actually performing the operations above indicated, it will be readily found, that

$$P = a^m + pa^{m-1} + qa^{m-2} + \ldots + sa^2 + ta + u.$$

Which value is obtained by barely substituting a for x in the equation first proposed.

And, by collecting the several terms of the coefficient of z, it will likewise appear, that

$$Q = ma^{m-1} + (m-1)pa^{m-2} + (m-2)qa^{m-3} + \ldots + 2sa + t.$$

Which last value is found by multiplying each of the terms of the former by the index of a in that term, and diminishing the same index by unity (r).

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lie between that pair of these numbers which give results that are the nearest to that in the question.

Thus, as an example of the latter case, let there be taken the equation

$$x^3 - 7x + 7 = 0.$$

Then, if 0, 1, 2, 3, &c. be substituted for x, the results will be

$$+7,\ +1,\ +1,\ +13,\ +43 \text{ \&c.}$$

All of which, after the third, deviate more and more from the true result.

Whence, as the roots of the equation are in this case all real, two of them must lie between 1 and 2.

And, if $1 + z$, or $2 - z$ be taken, as in the common rule, the values of x, after a certain number of operations, performed in the usual way, will be found to be 1.356895 and 1.692021, and the negative root is -3.048916.

(r) The method here pointed out, of deriving the coefficient of z from the value of P, is equally applicable to all the terms of the transformed equation; or to any polynomial,

$$x^m + px^{m-1} + qx^{m-2} + \ldots + sx^2 + tx + u,$$

which is to be converted into another, by substituting $x + z$ for x; or changing it to

Hence, since z, by hypothesis, is a small fraction, if the terms that involve its several powers, z^2, z^3, z^4, &c. which are all much smaller than z, be neglected in the transformed equation, we shall have $P + Qz = 0$, or

$$z = \frac{a^m + pa^{m-1} + \ldots + ta + u}{ma^{m-1} + (m-1)pa^{m-2} + \ldots + t}$$

And, consequently, if the numeral value of this expression be calculated to one or two places of decimals, and put equal to b, the first approximate part of the root will be

$$z = b, \text{ or } x = a + b = \alpha.$$

Whence, also, if this value of x, which is nearer its true value than the assumed number a, be substituted in the place of a in the above formula, it will become

$$z = \frac{\alpha^m + p\alpha^{m-1} + \ldots + t\alpha + u}{m\alpha^{m-1} + (m-1)p\alpha^{m-2} + \ldots + t}.$$

Which expression being now calculated to three or

$$(x+z)^m + p(x+z)^{m-1} + \ldots + s(x+z)^2 + t(x+z) + u.$$

For, if the several terms of this last expression be involved, and arranged according to the ascending powers of z, it can always be exhibited under the form

$$X + X'z + \frac{X''}{2}z^2 + \frac{X'''}{2.3}z^3 + \ldots + z^m.$$

Where the respective values of X, X', X'', &c. which are so many different functions of z, are

$$X = x^m + px^{m-1} + qx^{m-2} + \ldots + sx^2 + tx + u,$$
$$X' = mx^{m-1} + (m-1)px^{m-2} + (m-2)qx^{m-3} + \ldots + 2sx + t,$$
$$X'' = m(m-1)x^{m-2} + (m-1)(m-2)px^{m-3} + (m-2)(m-3)qx^{m-4} + \ldots + tq.$$

&c. &c.

Each of which, it is evident, may be derived from that immediately preceding it, by multiplying its several terms by the index of x in those terms, and diminishing the index by unity.

four places of decimals, and put equal to c; we shall have, for a second approximation towards the unknown part of the root,

$$z = c, \text{ or } x = a + c = \beta.$$

And if this value of x, which is still nearer the truth than the former, be again substituted for a, in the same formula, it will become

$$z = \frac{\beta^m + p\beta^{m-1} + \ldots + t\beta + u}{m\beta^{m-1} + (m-1)p\beta^{m-2} + \ldots + s}.$$

Where, if the right hand member be calculated to six or seven places of decimals, and put equal to d, we shall have, for a third approximation,

$$z = d, \text{ or } x = \beta + d = \gamma.$$

And, by proceeding in this manner, the approximation may be carried on to any assigned degree of exactness.

Any negative root of an equation may, also, be found in the same manner, by first changing the signs of all the alternate terms, as in Art. (N) 7, and then taking the positive roots of this equation, when determined as above, for the negative roots of the proposed equation.

It may likewise be observed, as a useful check upon this method of approximation, that the several values of z, in the successive steps of the operation, must continually decrease, or otherwise the value of x will not be rightly ascertained.

When the coefficients of the proposed equation are large numbers, the calculation may be generally abridged, by converting it into another, that shall have its coefficients smaller.

Thus, if it should be required to find a near approximate root of the equation

$$x^3 + 20000x - 5000000 = 0,$$

Let $x = 100z$; and we shall have, by substitution and division,

$$z^3 - 2z - 5 = 0;$$

Where, if the value of z be now found as before, that of x will be obtained by multiplying this number by 100.

And if, instead of the equation above given, we had

$$x^3 - 201x - 5123 = 0,$$

the substitution of $10z$ for x, or putting $x = 10z$, will give

$$z^3 - 2.01z - 5.123 = 0;$$

Where, since the roots differ but little from those of $z^3 - 2z - 5 = 0$, they may be calculated as for that equation, without any material error, and then multiplied by 10 for x.

The equation also, both in this and most other cases, may often be more readily resolved, by means of the form that has been given by Lagrange to this method of approximation, in his *Treatise de la Résolution des Equations Numériques*; which has, likewise, the peculiar advantage of showing the degree of accuracy obtained by every result, at each step of the operation (*s*).

(*s*) Lagrange, in the work here mentioned, as has been before observed, has also laid down a general method for determining the roots of numeral equations, in all cases where they are real, without its being necessary for us to have any previous knowledge of their nature or magnitude, which consists in finding a quantity, or the limit of a quantity, that shall be less than the least difference between any two roots of the given equation; and then, by means of what is called the equation of differences, and that

Thus, as a particular example of this method, let there be taken the last mentioned equation

$$x^3 - 2x - 5 = 0 \quad . \quad . \quad . \quad . \quad (1)$$

In which, having found, by trial, the nearest whole number, immediately below the root sought, to be 2; put

$$x = 2 + \frac{1}{y}, \text{ where } y = 1.$$

Then, if this value be substituted for x, in equation (1), the result, after clearing it of fractions, and changing the signs, will be

$$y^3 - 10y^2 - 6y - 1 = 0 \quad . \quad . \quad . \quad . \quad (2)$$

Where, having found, as before, the nearest integral value of y, below the true root, to be 10, we shall have, for a first approximation,

$$x = 2 + \frac{1}{10} = \frac{21}{10}.$$

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first proposed, to determine the roots, as above. But although the theory, which he has established for this purpose, is highly ingenious, and worthy of his great talents, yet the practical application of it is burdened with so many separate considerations and embarrassing operations, that, for equations of the higher orders, the labour becomes intolerable; which circumstance he appears to have been well aware of, as he has no where applied it to any equation above a cubic. It seems, indeed, that this operose mode of proceeding was chiefly intended to obviate what he considers as a great defect in Newton's method of approximation, above given, which is, that by the bare substitution of the numbers 0, 1, 2, 3, &c. for the unknown quantity, some of the roots of the equation, which differ from each other only by a small fraction, may be passed over without their being perceived; but this difficulty, which is only apparent, can be always avoided, by finding, as in the example to the last note, between which of those numbers the roots must lie, and then proceeding as usual.

And if, in conformity with the principle here used, we now put

$$y = 10 + \frac{1}{z}, \text{ where } z = 1,$$

and substitute this value for y in equation (2), there will arise, after clearing the result of fractions,

$$61z^3 - 94z^2 - 20z - 1 = 0 \quad . \; . \; . \; . \; (3)$$

From which having found, as in the former instances, the nearest integral value of z to be 1, we shall have, for a second approximation,

$$y = 11, \text{ and } x = 2 + \frac{1}{11} = \frac{23}{11}.$$

Also, by again taking, according to the same method,

$$z = 1 + \frac{1}{u}, \text{ where } u = 1.$$

And substituting this value for z in equation (3), there will arise

$$54u^3 + 25u^2 - 89u - 61 = 0.$$

Where the integral value of u being 1, we shall have, for a third approximation,

$$z = 2, \; y = 10 + \frac{1}{2}, \text{ and } x = 2 + \frac{1}{10 + \frac{1}{2}} = \frac{44}{21}.$$

And by proceeding in this way, through ten successive substitutions, the several values of the root, thus obtained, will be

$$\frac{2}{1}, \frac{21}{10}, \frac{23}{11}, \frac{44}{21}, \frac{111}{53}, \frac{155}{74}, \frac{576}{275}, \frac{731}{349}, \frac{1307}{624}, \frac{16415}{7837}$$

Which results are alternately less and greater than the true value of x; but each nearer the truth than that immediately preceding it.

So that, according to the doctrine of continued fractions, by which these results may be more rea-

dily derived, the last of them will differ from the root sought by less than

$$\frac{1}{(7837)^2} \text{ or } .0000000164.$$

Whence, by converting $\frac{16415}{7837}$ into decimals, we shall have

$$x = 2.09455148.$$

Which is correct, as far as to the last figure inclusively.

And if the equation that is to be resolved has more than one real root, they may be all determined, either by this method or the former, by first finding the limits between which they lie, and then proceeding as above (*t*).

(*t*) Various methods of approximating to the roots of equations, besides that of double position, given in the former part of the work, and those above mentioned, have been proposed by Simpson, p. 162 of his Algebra, and p. 215 et seq. of his Select Exercises, in which the number of figures in the root may be doubled, tripled, or even quadrupled at each operation.

But one of the most commodious formulæ of this kind, which I have yet seen, is that given by Mr. Barlow in the 12th number of Leybourn's Mathematical Repository; which is as follows:

Let $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + \&c. = w$, be any general equation, and let a be an approximate value of x, so that

$$a^n + pa^{n-1} + qa^{n-2} + ra^{n-3} + \&c. = v;$$

then will

$$a + \frac{2(w-v)a}{(n-1)w \text{ or } v + (n+1)a^n + (n-1)pa^{n-1} + (n-3)qa^{n-2} + \&c.}$$

be another approximate value of x, (using w, or v, according as a is integral or decimal,) which will, in all cases, at least double the number of figures in a; and in most of them the approximation will be considerably greater; so that by repeating the operation, with the last obtained value of x, the root may be speedily obtained to six or eight places of decimals.

This formula, for the solution of cubic equations, becomes

RECIPROCAL EQUATIONS.

(U) As the rules that have been given in Art (U), Part I, for resolving reciprocal equations, depend chiefly upon their being reducible to others of lower dimensions, it will be here proper to show how this reduction is effected, and the means it affords of obtaining the solutions there mentioned.

For this purpose, therefore, let there be taken the reciprocal equation

$$x^{2m}+px^{2m-1}+qx^{2m-2}+\ldots\ldots+qx^2+px+1=0,$$

which, when m is an integer, is always of an even number of dimensions.

Then, if this be divided by x^m, we shall have, by uniting the terms that are equally distant from the two extremes,

$$x^m+\frac{1}{x^m}+p\left(x^{m-1}+\frac{1}{x^{m-1}}\right)+q\left(x^{m-2}+\frac{1}{x^{m-2}}\right)+\&c.=0\,.\,(1)$$

Whence, putting $x+\frac{1}{x}=z$, the values of $x^2+\frac{1}{x^2}$, $x^3+\frac{1}{x^3}$, &c. as derived from each other, will be readily found to be as follows:

$$a+\frac{(w-v)a}{w \text{ or } v+2a^3+pa^2}$$

And for the solution of biquadratics,

$$a+\frac{2(w-v)a}{3w \text{ or } 3v+5a^4+3pa^3+qa^2-a}$$

And so on, for equations of the fifth, sixth, &c. power, the particular expressions for which are all easily deduced from the general formula.

$$x + \frac{1}{x} = z$$

$$x^2 + \frac{1}{x^2} = z^2 - 2$$

$$x^3 + \frac{1}{x^3} = z^3 - 3z$$

$$x^4 + \frac{1}{x^4} = z^4 - 4z^2 + 2$$

$$x^5 + \frac{1}{x^5} = z^5 - 5z^3 + 5z$$

&c. &c.

From the law of which expressions, when they are continued to the necessary extent, it will appear that

$$x^m + \frac{1}{x^m} = z^m - mz^{m-2} + \frac{m(m-3)}{2}z^{m-4} - \frac{m(m-4)(m-5)}{2 \cdot 3}z^{m-6}$$
$$+ \frac{m(m-5)(m-6)(m-7)}{2 \cdot 3 \cdot 4}z^{m-8} - \&c.$$

where it is to be observed, that, in taking the numeral values of m, the series is to be continued only as far as the powers of z are positive.

And as z, in this case, only rises to the mth power, it is plain that equation (1), when all its terms are taken according to their values in the general formula, and collected together, will only be of half the dimensions of the given equation, agreeably to the practical rule, Part I.

Again, let the proposed equation be of the form

$$x^{2m+1} + px^{2m+1} + qx^{2m-1} + \ldots + qx^2 + px + 1 = 0,$$

which, when m is an integer, is always of an odd number of dimensions.

Then, collecting the terms together, as in the former instance, we shall have

$$(x^{2m+1} + 1) + px(x^{2m-1} + 1) + qx^2(x^{2m-3} + 1) + \&c. = 0.$$

And if this be divided by $x+1$, which, by Art (E), Part I, can always be done, there will arise, by taking the terms of the several quotients of $x^{2m+1}+1$, $px(x^{2m}+1)$, &c. in their proper order,

$$x^{2m}-(1-p)x^{2m-1}+(1-p+q)x^{2m-2}-(1-p+q-r)$$
$$x^{2m-3}+\ldots+(1-p+q)x^2-(1-p)x+1=0,$$

which being a reciprocal equation one degree lower than the given one, and of an even number of dimensions, it is plain, that, like the former, it may be depressed to one half its present power.

Whence, in general, every reciprocal equation of the degree $2m$, as well as those of the degree $2m+1$, are always reducible to others of the mth degree.

Hence, in order to exemplify the rules before mentioned, let there be taken the equation

$$x^6+px^5+qx^4+rx^3+qx^2+px+1=0.$$

Then, by dividing by x^3, and uniting the terms that are at an equal distance from the two extremes, we shall have

$$(x^3+\frac{1}{x^3})+p(x^2+\frac{1}{x^2})+q(x+\frac{1}{x})+r=0.$$

And, consequently, by putting $x+\frac{1}{x}=z$, and substituting for $x+\frac{1}{x}$, $x^2+\frac{1}{x^2}$, and $x^3+\frac{1}{x^3}$, their values z, z^2-2, and z^3-3z, the equation will become

$$z^3+pz^2+(q-3)z+r-2p=0;$$

which being a cubic, if its three roots be respectively substituted for z in the quadratic equation

$$x+\frac{1}{x}=z, \text{ or}$$

$$x^2 - zx + 1 = 0,$$

we shall have, by finding the several corresponding values of x, all the six roots of the proposed equation.

Again, let there be taken the following equation of the fifth power,

$$x^5 + px^4 + qx^3 + qx^2 + px + 1 = 0.$$

Then, by collecting the terms, as in the general case above given, we shall have

$$(x^5 + 1) + px(x^3 + 1) + qx^2(x + 1) = 0;$$

which being divided, as is there pointed out, by $x + 1$, the quotient will become

$$x^4 - (1 - p)x^3 + (1 - p + q)x^2 - (1 - p)x + 1 = 0.$$

And, if this be again divided by x^2, the result, when the terms are properly united, will be

$$x^2 + \frac{1}{x^2} - (1 - p)(x + \frac{1}{x}) + 1 - p + q = 0.$$

Whence, substituting for $x + \frac{1}{x}$ and $x^2 + \frac{1}{x^2}$, their values z and $z^2 - 2$, as before, we shall have

$$z^2 + (p - 3)z + 1 - p + q = 0;$$

which being a quadratic, if its two roots be respectively substituted for z in the equation

$$x + \frac{1}{x} = z, \text{ or}$$

$$x^2 - zx + 1 = 0,$$

the four values of x, thus arising, and -1, which is always one of the roots of an equation of this kind, will be the five roots of the given equation.

And in the same manner may any other reciprocal equation be resolved, as far as to the ninth power inclusively; after which, the reduced equation will

be of the fifth order; which, in the present state of algebra, cannot be generally resolved.

BINOMIAL EQUATIONS.

(v) THE theory of reciprocal equations, laid down in the last Article, may be readily applied to the resolution of binomial equations, or such as are of the form

$$x^m \pm 1 = 0;$$

which, like those that consist of a greater number of terms, have always as many roots as there are units in the index m of the unknown quantity.

In order, therefore, to render the practical rules that are given in the first part of the work, more perspicuous, it will be proper to divide them into the four following cases.

1. When the equation is of an even number of dimensions, and has its second term positive, as

$$x^{2m} + 1 = 0.$$

This, by dividing each of its terms by x^m, will become

$$x^m + \frac{1}{x^m} = 0.$$

And, therefore, putting, as in the case of reciprocal equations,

$$x + \frac{1}{x} = z, \text{ or}$$

$$x^2 - zx + 1 = 0, \quad . \quad . \quad . \quad . \quad (1)$$

we shall have, by the general formula there given,

$$z^m - mz^{m-2} + \frac{m(m-3)}{2} z^{m-4} - \frac{m(m-4)(m-5)}{2 \cdot 3} z^{m-6} + \&c. = 0;$$

which, as in the former case, is to be continued only as far as the powers of z are positive.

Whence, finding the several values of z in this equation and substituting them successively for z, in equation (1), the roots of the proposed equation, may be readily obtained, as far as to the eighth power inclusively; being, in this case, all imaginary (*u*).

2. When the equation is of an odd number of dimensions, and has its second term positive, as

$$x^{2m+1} + 1 = 0.$$

This has, evidently, one of its roots $= -1$; and, therefore, by dividing it by $x+1$, we shall have

$$x^{2m} - x^{2m-1} + x^{2m-2} - \ldots + x^2 - x + 1 = 0.$$

And, by again dividing this last equation by x^m, and collecting the terms that are equally distant from the two extremes, there will arise

$$x^m + \frac{1}{x^m} - (x^{m-1} + \frac{1}{x^{m-1}}) + (x^{m-2} + \frac{1}{x^{m-2}}) - \&c. = 0.$$

Whence, putting, as in the former case, $x + \frac{1}{x} = z$, we shall have, by substituting the several values of $x^m + \frac{1}{x^m}$, $x^{m-1} + \frac{1}{x^{m-1}}$, &c. in the general formula above given,

$$z^m - z^{m-1} - (m-1)z^{m-2} + (m-2)z^{m-3} + (\frac{m^2}{2} - \frac{5m}{2} + 3)z^{m-4} \&c. = 0.$$

(*u*) After the eighth degree here mentioned, and the ninth and tenth in the following cases, the equation in z would rise to the fifth power, which, as before observed, cannot be resolved by any general algebraical formula yet known.

And, consequently, if each of the values of z, in this last equation, be successively substituted for z in the general equation

$$x^2 - zx + 1 = 0,$$

the several roots of the proposed equation will be obtained as far as to the ninth power inclusively; one of them being -1, and the other imaginary.

3. When the equation is of an even number of dimensions, and has its second term negative, as

$$x^{2m} - 1 = 0.$$

This has, evidently, two real roots, one of which is $=1$ and the other $=-1$; and, therefore, by dividing it by x^2-1, we shall have

$$x^{2m-2} + x^{2m-4} + x^{2m-6} + \ldots + x^4 + x^2 + 1 = 0.$$

And, by again dividing this last equation by x^{m-1}, and collecting the terms that are equally distant from the two extremes, there will arise

$$(x^{m-1} + \frac{1}{x^{m-1}}) + (x^{m-3} + \frac{1}{x^{m-3}}) + (x^{m-5} + \frac{1}{x^{m-5}})\&c. = 0.$$

Whence, putting, as before, $x + \frac{1}{x} = z$, we shall have, by substituting the several values of $x^{m-1} + \frac{1}{x^{m-1}}$, $x^{m-3} + \frac{1}{x^{m-3}}$, &c. in the general formula before given

$$z^{m-1} - (m-2)z^{m-3} + (\frac{m^2}{2} - \frac{7m}{2} + 6)z^{m-5} -$$
$$(\frac{m^3}{6} - \frac{5m^2}{2} + \frac{37m}{3} - 20)z^{m-7}\&c. = 0.$$

And, consequently, if each of the values of z, in this last equation, be successively substituted for z in the general equation

$$x^2 - zx + 1 = 0,$$

the several roots of the proposed equation may be obtained as far as to the tenth power inclusively; the two first, mentioned above, being real, and the rest imaginary.

4. When the equation is of an odd number of dimensions, and has its second term negative, as

$$x^{2m+1} - 1 = 0.$$

This has, evidently, one of its roots $= 1$; and, therefore, by dividing it by $x - 1$, we shall have

$$x^{2m} + x^{2m-1} + x^{2m-2} + \ldots + x^2 + x + 1 = 0.$$

And, by again dividing this last equation by x^m, and collecting the terms as before, there will arise

$$x^m + \frac{1}{x^m} + (x^{m-1} + \frac{1}{x^{m-1}}) + (x^{m-2} + \frac{1}{x^{m-2}}) + \&c. = 0.$$

Whence, putting, as in the former instances, $x + \frac{1}{x} = z$, we shall have, by substituting the several values of $x^m + \frac{1}{x^m}$, $x^{m-1} + \frac{1}{x^{m-1}}$, &c. in the general formula,

$$z^m + z^{m-1} - (m-1)z^{m-2} - (m-2)z^{m-3} + (\frac{m^2}{2} - \frac{5m}{2} + 3)z^{m-4} \&c. = 0.$$

And, therefore, if each of the values of z, in this equation, be substituted for z in the common equation

$$x^2 - zx + 1 = 0,$$

the roots of the proposed equation may be obtained as far as to the ninth power inclusively; the first of which is real, and the rest imaginary.

Hence, according to the above formulæ, the

several roots of 1 and -1, as far as to $\sqrt[4]{1}$ and $\sqrt[4]{-1}$, inclusively, may be exhibited as follows:

$$\sqrt{1} = +1, -1; \sqrt{-1}, = \sqrt{-1}$$
$$\sqrt[3]{1} = 1, -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-3}, -\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-3}$$
$$\sqrt[3]{-1} = -1, \tfrac{1}{2}+\tfrac{1}{2}\sqrt{-3}, \tfrac{1}{2}-\tfrac{1}{2}\sqrt{-3}$$
$$\sqrt[4]{1} = 1, -1, +\sqrt{-1}, -\sqrt{-1}$$
$$\sqrt[4]{-1} = \tfrac{1}{2}\sqrt{2}\pm\tfrac{1}{2}\sqrt{-2}, -\tfrac{1}{2}\sqrt{2}\pm\tfrac{1}{2}\sqrt{-2}$$

It may here, also, be observed, that if the index of the unknown quantity, in any binomial equation, be a composite number, as in the case

$$x^{mn} - 1 = 0,$$

the roots of every such equation may be always determined by the resolution of two or more similar equations of the forms

$$x^m - 1 = 0, \text{ and } x^n - 1 = 0.$$

For x^m and x^n being, here, each $= 1$, it is evident, that $x^{mn} = 1$, or $x^{mn} - 1 = 0$; and, consequently, the products of every two corresponding roots of the latter equations will be roots of the former (x).

Hence, since it appears, from what has been before shown, that the roots of the equations

$$x^2 - 1 = 0, \; x^3 - 1 = 0, \; x^5 - 1 = 0,$$

may be obtained solely by means of a quadratic equation, it follows, that we can always resolve any binomial equation

$$x^m - 1 = 0,$$

(x) When each of the two members of the equation $x^m = 1$ are elevated to the nth power, it is plain that we introduce $m(n-1)$ values of x, which are foreign to the proposed equation; for $x^m = 1$ has only m roots, and $x^{mn} = 1$ has mn roots; of which the difference is $mn - m$; and the same is true of the equation $x^m = -1$.

when it has no other simple factors than 2, 3, and 5, or when it is of the form

$$2^{\lambda} \times 3^{\mu} \times 5^{\nu}.$$

And, in admitting the resolution of equations of the third degree, we can determine the roots of the equation

$$x^7 - 1 = 0,$$

and, consequently, also, those of the above mentioned equation

$$x^m - 1 = 0,$$

when the index m has no other simple factors than 2, 3, 5, 7; or when it is of the form

$$2^{\lambda} \times 3^{\mu} \times 5^{\nu} \times 7^{\pi}.$$

Some particular cases of binomial equations may likewise be resolved directly, without having recourse to the equation in z.

Thus, if the equation $x^3 \pm 1 = 0$, or $x^3 = \mp 1$, which has -1 or $+1$ for one of its roots, according as the second term is $+$ or $-$, be divided by $x \pm 1$, the quotient will be

$$x^2 \mp x + 1 = 0.$$

From which the other two roots will be found to be $\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$, or $-\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$, as above.

In like manner, the equation $x^4 - 1 = 0$, or $x^4 = 1$, gives $x^2 = \pm 1$, or $x = \pm\sqrt{\pm 1}$; and $x^5 - 1 = 0$ gives $x^5 = \pm 1$, the roots of which can be found as in the first case.

Also, since the equation $x^8 - 1 = 0$, is the product of $x^4 + 1$ and $x^4 - 1$, we shall have, by putting $x^4 + 1 = 0$ and $x^4 - 1 = 0$,

$$x^2 = \pm 1, \text{ or } x = \pm\sqrt{\pm 1}; \text{ and}$$

$$x^2 = \pm\sqrt{-1}, \text{ or } x = \pm\sqrt{(\pm\sqrt{-1})}:$$

which are the eight roots or values of x in the given equation.

And if the proposed equation be $x^8 - 256 = 0$, where $x = \sqrt[8]{256} = 2$, its eight roots will be

$$\pm 2\sqrt{\pm 1}, \text{ and } \pm 2\sqrt{(\pm\sqrt{-1})}\ (y).$$

In general, when m is an even number, the equation

$$x^m - 1 = 0, \text{ or } x^m = 1,$$

has two real roots, $+1$ and -1; and when m is an odd number, it has only one real root, which is $+1$; the rest of the roots in each of these cases being imaginary.

In like manner, if m be an odd number, the equation

$$x^m + 1 = 0, \text{ or } x^m = -1,$$

has only one real root, which is -1, the rest being imaginary; and when m is an even number, it has no real root, because $\sqrt[m]{-1}$, in that case, is imaginary.

It may, also, be farther observed, that the imaginary roots of any binomial equation, in which the index m, of the unknown quantity, is a prime number, have the remarkable property of being all of the form

$$\alpha,\ \alpha^2,\ \alpha^3,\ \alpha^4,\ \&c.$$

(y) These examples are sufficient to show that every equation of the form

$$x^m - a = 0, \text{ or } x^m + a = 0,$$

has as many roots, real and imaginary, as there are units in m; and though the latter of these cannot be considered as real quantities, but merely as pure algebraical symbols, of no determinate value, yet when they are submitted to the operations indicated by the equation, the two members of that equation will be always identical.

Thus, let there be taken, as a partial example of this kind, the equation

$$x^3 - 1 = 0, \text{ or } x^3 = 1;$$

of which the three roots, found as above, are

$$1, \ -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-3}, \text{ and } -\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-3}.$$

Then, putting $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}=\alpha$, and $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}=\beta$, we shall have, by the theory of equations, Art. (M) 3,

$$1 \times \alpha \times \beta = 1, \text{ or } \alpha\beta = 1.$$

And, consequently, since $\alpha^3=1$, and $\beta^3=1$, there will arise, by multiplying the last equation, $\alpha\beta=1$, by α^2, or β^2,

$$\alpha^3\beta=\alpha^2, \text{ and } \beta^3\alpha=\beta^2; \text{ or } \beta=\alpha^2, \text{ and } \alpha=\beta^2.$$

Hence it follows, that either of the two imaginary roots, in this case, is equal to the square of the other.

It likewise appears from the equation $\alpha\beta=1$, and the one last given, that

$$\alpha=\frac{1}{\beta}=\frac{1}{\alpha^2}, \text{ and } \beta=\frac{1}{\alpha}=\frac{1}{\beta^2}.$$

So that the same two imaginary roots are, also, the reciprocals of each other.

And if there be taken the other case of this equation

$$x^3 + 1 = 0, \text{ or } x^3 = -1,$$

where the three roots, found as before, are

$$-1, \ \tfrac{1}{2}+\tfrac{1}{2}\sqrt{-3}, \text{ and } \tfrac{1}{2}-\tfrac{1}{2}\sqrt{-3},$$

we shall have, by putting $\frac{1}{2}+\frac{1}{2}\sqrt{-3}=\alpha$, and $\frac{1}{2}-\frac{1}{2}\sqrt{-3}=\beta$, as in the former instance,

$$\alpha=-\beta^2 \text{ and } \beta=-\alpha^2; \text{ or } \alpha=-\frac{1}{\alpha^2}, \text{ and } \beta=-\frac{1}{\beta^2}.$$

Hence the three roots, or values of x, in the equation

$$x^3 = 2, \text{ are } \sqrt[3]{2},\ \alpha\sqrt[3]{2}, \text{ and } \alpha^2\sqrt[3]{2}.$$

And the three roots, or values of x, in the equation

$$x^3 = -b^3, \text{ are } -b,\ -\alpha b, \text{ and } -\alpha^2 b.$$

Or, independently of this method of proof, it may be shown, in a more general way, that all the roots, both real and imaginary, of the equation

$$x^m - 1 = 0, \text{ or } x^m = 1,$$

when m is a prime number, are different powers of the same quantity.

For let α represent any one of the imaginary roots; then, since $\alpha^m = 1$, we shall, also, have

$$\alpha^{2m} = 1,\ \alpha^{3m} = 1,\ \alpha^{4m} = 1,\ \alpha^{5m} = 1, \text{ \&c.}$$

And, consequently, if α be a root of the given equation, so likewise will every term in the series,

$$\alpha,\ \alpha^2,\ \alpha^3,\ \alpha^4, \ldots\ldots\ \alpha^m, \text{ or } 1.$$

Also, supposing m, as above, to be a prime number, and α to be one of the roots of the equation

$$x^m + 1 = 0, \text{ or } x^m = -1,$$

the several roots, in this case, will be found, by a similar mode of reasoning, to be

$$-\alpha,\ -\alpha^2,\ -\alpha^3,\ -\alpha^4, \ldots\ldots\ -\alpha^m, \text{ or } -1;$$

where it may likewise be observed, that any one of the roots, in each of these cases, may be represented by $\pm\alpha^n$, provided n be not a multiple of m.

Thus, for example, taking, in the first case, $m = 3$, the roots will be α, α^4, and α^3, or 1; and if α^2 be put in the place of α, we shall have the three roots α^2, α^4, α^6; which,

on account of α^5 being $=1$, become α^2, α, and α^3, as before.

Also, when $m=5$, the roots will be α, α^2, α^3, α^4, and α^5, or 1; and if α^2 be substituted for α, and then raised to the first, second, third, fourth, and fifth power, we shall have α^2, α^4, α^6, α^8, α^{10}; which, from α^5 being $=1$, will become α^2, α^4, α, α^3, and α^5.

And the same results will be obtained, by substituting α^3, or α^4, in the place of α; and so on for any other case of this kind.

Hence, likewise, because $\alpha^m=1$, we shall have, by the nature of powers,

$$\alpha^{m-1}=\frac{\alpha^m}{\alpha}=\frac{1}{\alpha},\ \alpha^{m-2}=\frac{\alpha^m}{\alpha^2}=\frac{1}{\alpha^2}\ \ldots\ \alpha=\frac{\alpha^m}{\alpha^{m-1}}=\frac{1}{\alpha^{m-1}},\text{ or}$$
$$\frac{1}{\alpha},\ \frac{1}{\alpha^2},\ \frac{1}{\alpha^3},\ \frac{1}{\alpha^4}\ \ldots\ldots\ \frac{1}{\alpha^{m-1}};$$

which are, also, the several roots of the same equation, except 1, taken in an inverse order.

And, consequently, if these expressions be substituted for s_m, in Art. (N) 4, the sums of the several powers of the roots will be

$$1+\alpha^n+\alpha^{2n}+\ \ldots\ldots\ +\alpha^{(n'-1)n}=n',\text{ or } 0;$$
$$1+\frac{1}{\alpha^n}+\frac{1}{\alpha^{2n}}+\ \ldots\ldots\ +\frac{1}{\alpha^{(n'-1)n}}=n',\text{ or } 0;$$

according as n is a multiple or not a multiple of n' (z).

(z) The principles above given are equally applicable to all such equations as are reducible to those of the second degree, as

$$x^{2m}-px^m+q=0.$$

For, since, by the usual method of solving quadratic equations, we have

$$x^m=\tfrac{1}{2}p\pm\sqrt{(\tfrac{1}{4}p^2-q)};$$

But as the analytical method of solution, above given, is confined within certain limits, I shall now show, in as concise a manner as possible, how the roots of any binomial equation of the form

$$x^m - 1 = 0, \text{ or } x^m + 1 = 0,$$

may be obtained universally, by means of the division of the circle into m equal parts.

For this purpose, let z denote any arc, and π the semicircumference of the circle, of which the radius is 1.

Then, by Art. 69, p. 334, of my Trigonometry, second edition, we shall have the following remarkable property,

$$(\cos. z \pm \sin. z\sqrt{-1})^m = \cos. mz \pm \sin. mz\sqrt{-1};$$

which expression, by barely substituting $\frac{z}{m}$ instead of z, will become of the form

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if these two values be put equal to α and β, the question will be reduced to finding the roots of the binomial equations

$$x^m = \alpha \text{ and } x^m = \beta.$$

Thus, suppose it were required to find all the six roots of the equation

$$x^6 - 35x^3 + 216 = 0,$$

we shall have, by extracting the root according to the method above mentioned,

$$x^3 = 8, \text{ and } x^3 = 27.$$

Where the rational values of x being 2 and 3, the six roots of the proposed equation will be

$$2,\ 2\alpha,\ 2\alpha^2,\ 3,\ 3\alpha, \text{ and } 3\alpha^2.$$

And if it were required to determine the eight roots of the equation

$$x^8 + 65x^4 - 1296 = 0,$$

they will be found, by following a similar method, to be

$$\pm 2,\ \pm 2\sqrt{-1},\ \pm 3\sqrt{(+\sqrt{-1})}, \text{ and } \pm 3\sqrt{(-\sqrt{-1})}.$$

$$(\cos.\frac{z}{m} \pm \sin.\frac{z}{m}\sqrt{-1})^m = \cos. z \pm \sin. z\sqrt{-1}.$$

And, consequently, supposing x to be represented by $\cos.\frac{z}{m} \pm \sin.\frac{z}{m}\sqrt{-1}$, and z, to be so taken that $\cos. z \pm \sin. z\sqrt{-1}$ shall be $=1$, this last equation will be similar to the equation

$$x^m - 1 = 0, \text{ or } x^m = 1.$$

But the cosine of any multiple of the circumference is $=1$, and the sine of the same multiple $=0$, (Trig. Art. 13, p. 293); whence, putting $z = n \times 2\pi$, we shall have

$$(\cos.\frac{n}{m}2\pi \pm \sin.\frac{n}{m}2\pi\sqrt{-1})^m = \cos.n2\pi \pm \sin.n2\pi\sqrt{-1}.$$

Or, taking the quantities, which, according to the above hypothesis, have been assumed equal to them,

$$x^m = \cos. n2\pi \pm \sin. n2\pi\sqrt{-1}, \text{ and}$$

$$x = \cos.\frac{n}{m}2\pi \pm \sin.\frac{n}{m}2\pi\sqrt{-1};$$

where it is plain, that each numeral value of n, in the latter, will give two different values of x.

If m, in this case, be any even number $2k$, the last formula will become

$$x = \cos.\frac{n}{k}\pi \pm \sin.\frac{n}{k}\pi.$$

In which, substituting 0, 1, 2, 3, 4, ... k, successively for n, we shall have,

For
$$\left|\begin{array}{ll} n=0, & x=\cos.0 \pm \sin.0\sqrt{-1} = +1 \\ n=1, & x=\cos.\frac{\pi}{k} \pm \sin.\frac{\pi}{k}\sqrt{-1} \\ \cdots & \cdots\cdots\cdots\cdots \\ n=k-1, & x=\cos.\frac{k-1}{k}\pi \pm \sin.\frac{k-1}{k}\pi\sqrt{-1} \\ n=k, & x=\cos.\frac{k}{k}\pi \pm \sin.\frac{k}{k}\pi\sqrt{-1} = -1 \end{array}\right.$$

where $2k-2$, or $m-2$, values of x are imaginary, and two real, which are all the question admits of; as the substitution of any other number for n would occasion these values to return upon themselves.

When m is any odd number, $2k+1$, the same formula will become

$$x = \cos.\frac{2n}{2k+1}\pi \pm \sin.\frac{2n}{2k+1}\pi\sqrt{-1}.$$

In which, taking $n=0, 1, 2, 3, 4, \ldots\ldots k$, successively for n, as before, we shall have

$$\text{For}\left|\begin{array}{l} n=0,\ x=\cos.0 \pm \sin.0\sqrt{-1} = +1 \\ n=1,\ x=\cos.\frac{2}{2k+1}\pi \pm \sin.\frac{2}{2k+1}\pi\sqrt{-1} \\ n=2,\ x=\cos.\frac{4}{2k+1}\pi \pm \sin.\frac{4}{2k+1}\pi\sqrt{-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ n=k,\ x=\cos.\frac{2k}{2k+1}\pi \pm \sin.\frac{2k}{2k+1}\pi\sqrt{-1} \end{array}\right.$$

where $2k$, or $m-1$ values of x are imaginary and one real; which, for the reason given above, are all the question admits of.

A similar mode of proceeding will also enable us to resolve the equation

$$x^m + 1 = 0, \text{ or } x^m = -1,$$

by observing that the sine of any odd multiple of π is $=0$, and its cosine $=-1$, which, therefore, gives, in this case,

$$x^m = \cos.(2n+1)\pi \mp \sin.(2n+1)\pi\sqrt{-1},$$

and

$$x = \cos.\frac{2n+1}{m}\pi \mp \sin.\frac{2n+1}{m}\pi\sqrt{-1}.$$

Where each value of n, in the latter, furnishes two values of x, as before.

If m be an even number $2k$, this formula will become

$$x = \cos.\frac{2n+1}{2k}\pi \mp \sin.\frac{2n+1}{2k}\pi\sqrt{-1}.$$

In which, substituting 0, 1, 2, 3, 4 . . . . $k-1$, successively for n, we shall have,

$$\text{For}\left|\begin{array}{l} n=0,\ x=\cos.\frac{1}{2k}\pi \mp \sin.\frac{1}{2k}\pi\sqrt{-1} \\ n=1,\ x=\cos.\frac{3}{2k}\pi \mp \sin.\frac{3}{2k}\pi\sqrt{-1} \\ n=2,\ x=\cos.\frac{5}{2k}\pi \mp \sin.\frac{5}{2k}\pi\sqrt{-1} \\ \ldots\ldots\ldots\ldots\ldots \\ n=k-1,\ x=\cos.\frac{2k+1}{2k}\pi \mp \sin.\frac{2k+1}{2k}\pi\sqrt{-1} \end{array}\right.$$

Where all the m values of x are imaginary.

When m is an odd number $2k+1$, the same formula will become

$$x = \cos.\frac{2n+1}{2k+1}\pi \mp \sin.\frac{2n+1}{2k+1}\pi\sqrt{-1}.$$

In which, taking $n=0$, 1, 2, 3, ... k respectively, as above, we shall have,

$$\text{For}\left|\begin{array}{l} n=0,\ x=\cos.\frac{1}{2k+1} \mp \sin.\frac{1}{2k+1}\pi\sqrt{-1} \\ n=1,\ x=\cos.\frac{3}{2k+1} \mp \sin.\frac{3}{2k+1}\pi\sqrt{-1} \\ n=2,\ x=\cos.\frac{5}{2k+1} \mp \sin.\frac{5}{2k+1}\pi\sqrt{-1} \\ \ldots\ldots\ldots\ldots\ldots \\ n=k,\ x=\cos.\pi \mp \sin.\pi\sqrt{-1} = -1 \end{array}\right.$$

where $m-1$ values of x are imaginary, and one real; which, in this case, as well as in the former, are all that the question admits of.

It may here, also, be farther added, that when m is an even number $2k$, the binomial

$$x^m - 1, \text{ or } x^{2k} - 1,$$

may be decomposed into k real factors of the second degree.

And if m be an odd number, $2k+1$, the same binomial

$$x^m - 1, \text{ or } x^{2k+1} - 1,$$

will be the product of $(x-1)$ by k real factors of the second degree.

For, if the two roots $x=\alpha$, and $x=\beta$, reduce $x^m - 1$ to zero, the whole binomial will be divisible by $(x-\alpha)(x-\beta)$, or

$$x^2 - (\alpha+\beta)x + \alpha\beta.$$

But the values of x, as has been before shown, that will reduce $x^m - 1$ to 0, are

$$\cos.\frac{n}{m}2\pi + \sin.\frac{n}{m}2\pi\sqrt{-1}, \text{ and } \cos.\frac{n}{m}2\pi - \sin.\frac{n}{m}2\pi\sqrt{-1}.$$

Whence, by the addition and multiplication of these two formulæ, we shall have,

$$\alpha+\beta = 2\cos.\frac{n}{m}2\pi, \text{ and } \alpha\beta = 1.$$

And consequently the general expression of the factors of the binomial $x^m - 1$, will, in this case, be

$$x^2 - 2x\cos.\frac{n}{m}2\pi + 1.$$

Where it is evident that n must be so taken as not to be divisible by m, or otherwise the sign of $\frac{n}{m}2\pi$, in the above two forms, would be $=0$, and consequently α and β could not be different roots.

Hence, since x, in the binomial $x^{2k-1} - 1$, has two real values, $+1$ and -1, one of its factors will be $x^2 - 1$; and, consequently, the rest will be found by giving to n such of the values $1, 2, 3, \ldots k-1$, as will answer the conditions last mentioned.

Also, because one of the values of x in the binomial $x^{2k+1}-1$, is $+1$, it will have $x-1$ for one of its factors, and the rest will be determined, as before, by taking the proper values of n from the series 1, 2, 3 . . . . k.

Thus, agreeably to what is here laid down, we shall have, when m is an even number;

$$x^m-1=(x^2-1)(x^2-2x\cos.\frac{2\pi}{m}+)(x^2-2x\cos.\frac{4\pi}{m}+1)$$
$$(x^2-2x\cos.\frac{6\pi}{m}+1)\&c.$$

And when, as in the second case, m is an odd number, we shall have

$$x^m-1=(x-1)(x^2-2x\cos.\frac{2\pi}{m}+1)(x^2-2x\cos.\frac{4\pi}{m}+1)$$
$$(x^2-2x\cos.\frac{6\pi}{m}+1)\&c.$$

Whence, taking, by way of illustration, the following practical examples, it will readily appear, that

$$x^6-1=(x^2-1)(x^2-2x\cos.60^\circ+1)$$
$$(x^2-2x\cos.120^\circ+1),$$
$$x^5-1=(x-1)(x^2-2x\cos.72^\circ+1)$$
$$(x^2-2x\cos.144^\circ+1).$$

Where each factor, except $x-1$ in the last, will give two values of x.

Also, if there be taken the binomial x^m+1, in which, as before shown, we have

$$x=\cos.\frac{2n+1}{m}\pi\mp\sin.\frac{2n+1}{m}\pi\sqrt{-1},$$

the general expression of the factors, in this case, found, by combining the two values of x, as before, will be

$$x^2-2x\cos.\frac{2n+1}{m}\pi+1:$$

And, consequently, by taking n so that $2n+1$ shall not be divisible by m, we shall have the two formulæ,

$$x^m+1=(x^2-2x\cos.\frac{\pi}{m}+1)(x^2-2x\cos.\frac{3\pi}{m}+1)$$
$$(x^2-2x\cos.\frac{5\pi}{m}+1)\ \&c.$$

$$x^m+1=(x+1)(x^2-2x\cos.\frac{\pi}{m}+1)(x^2-2x\cos.\frac{3\pi}{m}+1)$$
$$(x^2-2x\cos.\frac{5\pi}{m}+1)\ \&c.$$

In the first of which m is supposed to be an even number, and in the last an odd number.

Whence it will also readily appear, as in the former instance, that

$$x^6+1=(x^2+1)(x^2-2x\cos.30^\circ+1)$$
$$(x^2-2x\cos.150^\circ+1)$$
$$x^5+1=(x+1)(x^2-2x\cos.36^\circ+1)$$
$$(x^2-2\cos.108^\circ+1)$$

Where each of the factors, except $x+1$ in the last, will furnish two values of x, as before.

In like manner, it may also be shown, that any equation, of two terms, of the form

$$y^m-a^m=0,\ \text{or}\ y^m+a^m=0,$$

to which they can all be reduced, is likewise decomposable into real factors of the first and second degree.

For let $y=ax$, or $x=\frac{y}{a}$; then, if this value be substituted for x, in each of the preceding equations, we shall have, by multiplying both sides of them by a^m, the four following formulæ:

$$1.\ y^m - a^m = (y^2 - a^2)(y^2 - 2ay\cos.\frac{2\pi}{m} + a^2)$$
$$(y^2 - 2ay\cos.\frac{4\pi}{m} + a^2)\&c.$$

$$2.\ y^m - a^m = (y - a)(y^2 - 2ay\cos.\frac{2\pi}{m} + a^2)$$
$$(y^2 - 2ay\cos.\frac{4\pi}{m} + a^2)\&c.$$

$$3.\ y^m + a^m = (y^2 - 2ay\cos.\frac{\pi}{m} + a^2)(y^2 - 2ay$$
$$\cos.\frac{3\pi}{m} + a^2)\&c.$$

$$4.\ y^m + a^m = (y + a)(y^2 - 2ay\cos.\frac{\pi}{m} + a^2)$$
$$(y^2 - 2ay\cos.\frac{3\pi}{m} + a^2)\&c.$$

In the first and third of which m is supposed to be an even number, and in the second and fourth an odd number (*a*).

(*a*) These formulæ, which are derived solely from the theory of binomial equations here treated of, are the analytical expressions for the celebrated property of the circle, known by the name of the *Cotesian Theorem*; and the factors of the before mentioned equation

$$x^{2m} - px^m + q = 0,$$

which can be determined in a similar manner, give the theorem of *De Moivre*; of which that of Cotes is only a particular case.

The method, also, which has lately been established by *Gauss*, in his *Disquisitiones Arithmeticæ*, of dividing the circumference of a circle into 17, 257, 65537, &c. equal parts, by means of quadratic equations only, is derived from the same principles, and does not seem capable of being effected in any other way. See the *Essai* of Legendre, before referred to, 2d edit., or Mr. Barlow's Elementary Investigation of the Theory of Numbers, where this curious subject is clearly explained and elucidated.

OF EQUATIONS THAT HAVE EQUAL ROOTS.

(U) As the methods usually employed for determining the equal roots of equations are, in general, extremely complicated, when applied to those of the fifth, sixth, and other higher dimensions, it will here only be necessary to consider the rules that have been laid down in the former part of the work, for equations of the first four degrees; which may be investigated as follows:

1. If a quadratic equation have two equal roots, with the same sign, it must evidently be of the form

$$x^2 \pm 2px + p^2 = 0;$$

where each of the roots is $-p$, or $+p$, according as the second term of the equation is positive or negative.

And if the roots have contrary signs, the equation will be of the form

$$x^2 - p^2 = 0, \text{ or } x^2 = p^2;$$

where the two values of x are manifestly $+p$ and $-p$.

2. If the proposed equation be a cubic, of the general form

$$x^3 + px^2 + qx + r = 0.$$

Let its three roots, in this case, be denoted by a, b, c.

Then, by what has been shown in a former article, we shall have the two following equations:

$$a^3 + pa^2 + qa + r = 0,$$
$$b^3 + pb^2 + qb + r = 0.$$

And, consequently, if the latter of these be subtracted from the former, there will arise

$$a^3 - b^3 + p(a^2 - b^2) + q(a - b) = 0.$$

Which last equation, when divided by $a - b$, will give

$$a^2 + ab + b^2 + p(a + b) + q = 0.$$

Whence, if we now take $a = b$, the reduced equation will be

$$3a^2 + 2pa + q = 0.$$

Or, by putting x for its equal a, this equation will become

$$3x^2 + 2px + q = 0;$$

where one of the values of x, taken twice with the same sign, will give the two equal roots of the proposed equation.

And since, from the composition of equations, we have $2a + b = -p$, the remaining root will be

$$x = -p - 2a.$$

When $p = 0$, the general equation, in that case, becomes

$$x^3 + qx + r = 0.$$

And if this has two equal roots, with the same sign, they will evidently be each $= \sqrt{-\frac{q}{3}}$; and the remaining root will be $= 2\sqrt{-\frac{q}{3}}$.

Again, if $+a$ and $-a$ be two roots of the proposed equation, and c the remaining root, one of its factors will manifestly be

$$x^2 - a^2.$$

And, consequently, if this be multiplied by $x + c$, it will readily appear that the given equation must always be of the form

$$x^3 + cx^2 - a^2x - a^2c = 0.$$

Whence, putting $c=p$, $-a^2=q$, and $-a^2c=r$, we shall have $-a^2c=pq$ and $-a^2c=r$, or

$$pq=r;$$

which is, therefore, the relation of the coefficients, when the equation has two equal roots, with contrary signs.

And, since $a^2c=-r$ and $c=p$, the two equal roots, in this case, will be

$$\sqrt{-\frac{r}{p}}, \text{ and } -\sqrt{-\frac{r}{p}},$$

and the third, or remaining root, will be p; which is the coefficient of the second term of the equation.

Hence, as each of these expressions will become infinite, when $p=0$, it is plain, that a cubic equation of the form

$$x^3+qx+r=0,$$

cannot have two equal roots with contrary signs.

3. If the proposed equation be a biquadratic, of the general form

$$x^4+px^3+qx^2+rx+s=0.$$

Let its four roots be denoted by a, b, c, and d.

Then, as in the former instance, we shall have the two following equations:

$$a^4+pa^3+qa^2+ra+s=0,$$
$$b^4+pb^3+qb^2+rb+s=0.$$

And, consequently, if the latter of these be subtracted from the former, there will arise

$$a^4-b^4+p(a^3-b^3)+q(a^2-b^2)+r(a-b)=0.$$

Which last equation, when divided by $a-b$, gives

$$a^3+a^2b+ab^2+b^3+p(a^2+ab+b^2)+q(a+b)+r=0.$$

Whence, a and b being each taken equal to x, as before, the reduced equation will be

$$4x^3 + 3px^2 + 2qx + r = 0.$$

And if this be multiplied by $x + \frac{1}{4}p$, in order to make it of a similar form with the given equation, we shall have

$$4x^4 + 4px^3 + (2q + \frac{3}{4}p^2)x^2 + (r + \frac{1}{2}pq)x + \frac{1}{4}pr = 0.$$

From which, if there be taken four times the given equation, the result, when multiplied by 4, will be

$$(3p^2 - 8q)x^2 + 2(pq - 6r)x + rp - 16s = 0;$$

where one of the values of x, taken twice with the same sign, will give the two equal roots of the proposed equation.

And if one of these equal roots be put $= \alpha$, the quotient arising from dividing the proposed equation by $x^2 - 2\alpha x + \alpha^2$ will be

$$x^2 + (p + 2\alpha)x + (q + 2\alpha p + 3\alpha^2) = 0;$$

which will give the other two roots of the same equation.

If $p = 0$, or, which is the same thing, if the given equation be of the form

$$x^4 + qx^2 + rx + s = 0.$$

The first of the above quadratics will become, by changing the signs of all its terms, and dividing by 4,

$$2qx^2 + 3rx + 4s = 0.$$

And, consequently, if one of the equal roots of this be denoted by α, as above, the other quadratic will be

$$x^2 + 2\alpha x + (q + 3\alpha^2) = 0.$$

Again, if $+a$ and $-a$ be two roots of the pro-

posed equation, and c, d, the remaining roots, its factors will evidently be

$$x^2 - a^2 \text{ and } x^2 + (b+c)x + bc.$$

And, therefore, by actually multiplying them, it is plain that the given equation must always be of the form

$$x^4 + (b+c)x^3 + (bc - a^2)x^2 - a^2(b+c)x - a^2bc = 0.$$

Whence, putting $b + c = p$, $bc - a^2 = q$, $-a^2(b+c) = r$, and $-a^2bc = s$, we shall readily obtain, by first eliminating a^2, and then b, c, the following equation

$$p(qr - ps) = r^2,$$

which expresses the relation of the coefficients, when the given equation has two equal roots, with contrary signs.

And since $b + c = p$ and $a^2(b+c)$, or $a^2p = -r$, the two equal roots, in this case, will be, as in that for the cubic equation,

$$+\sqrt{-\frac{r}{p}} \text{ and } -\sqrt{-\frac{r}{p}}.$$

Also, because $b + c = p$ and $bc = q + a^2$, if a, which is one of the above equal roots, be put $= \alpha$, the factor $x^2 + (b+c)x + bc$ will become

$$x^2 + px + q + \alpha^2 = 0;$$

which will give the other two roots of the proposed equation.

Where it is evident, as in the corresponding case of the cubic equation above mentioned, that a biquadratic of the form

$$x^4 + qx^2 + rx + s = 0,$$

cannot have two equal roots, with contrary signs.

4. If three of the four roots a, b, c, d, of the same general biquadratic equation

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

be equal to each other, and have the same sign; we shall have, by substituting a and b for x, as before,

$$a^4 + pa^3 + qa^2 + ra + s = 0,$$
$$b^4 + pb^3 + qb^2 + rb + s = 0.$$

And, consequently, by taking the latter of these equations from the former, there will arise,

$$a^4 - b^4 + p(a^3 - b^3) + q(a^2 - b^2) + r(a - b) = 0.$$

Which last result, being divided by $a - b$, gives

$$a^3 + a^2b + ab^2 + b^3 + p(a^2 + ab + b^2) + q(a + b) + r = 0.$$

Whence, supposing $a = b$, the reduced equation, in the first instance, will be

$$4a^3 + 3pa^2 + 2qa + r = 0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In like manner, if b and c be each substituted for x, we shall have

$$b^4 + pb^3 + qb^2 + rb + s = 0,$$
$$c^4 + pc^3 + qc^2 + rc + s = 0.$$

And, consequently, by subtracting the latter of these equations from the former, there will arise

$$b^4 - c^4 + p(b^3 - c^3) + q(b^2 - c^2) + r(b - c) = 0.$$

Which last result, being divided by $b - c$, gives

$$b^3 + b^2c + bc^2 + c^3 + p(b^2 + bc + c^2) + q(b + c) + r = 0.$$

Whence, supposing $b = c$, the reduced equation, in this case, will be

$$4b^3 + 3pb^2 + 2qb + r = 0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

And, consequently, if equation (2) be taken from equation (1) there will remain

$$4(a^2 - b^3) + 3p(a^2 - b^2) + 2q(a - b) = 0.$$

Which last result, being again divided by $a-b$, gives

$$4(a^2+ab+b^2)+3p(a+b)+2q=0.$$

Where, since a and b are each equal to x, we shall have, by substitution and dividing by 2,

$$6x^2+3px+q=0.$$

In which equation, one of the values of x, taken thrice, with the same sign, gives the three equal roots of the proposed equation.

And if $p=0$, or the equation be of the form

$$x^4+qx^2+rx+s=0,$$

the three equal roots will be each expressed by $\sqrt{-\frac{q}{6}}$.

Also, if $+a$, $+a$, $-a$, be the three equal roots, and b the remaining root, we shall have, by putting $c=b$, in the second part of case 3,

$$p(qr-ps)=r^2 \text{ and } x=\pm\sqrt{-\frac{r}{p}}.$$

Where it appears that the relation of the coefficients, and the values of the equal roots, are the same in both these instances (*b*).

OF EXPONENTIAL EQUATIONS.

(x) Many attempts have been made to determine the value of the unknown quantity, in the exponential equation $x^x=a$, by converting it into a series

(*b*) The unknown quantity, in all these cases, might have been easily found by means of a simple equation; but as the expression for the roots, when so determined, would, in general, be very complicated, it was judged better to reduce them only to quadratics, the solution of which is nearly as readily obtained as that of the former.

that shall consist only of a and its powers; but no expression of this kind has hitherto been discovered, that is sufficiently convergent to answer any practical purpose.

A very simple approximating rule, however, may be readily derived in this way, as follows:

Find, by trial, a number r, as near the true value of x as possible, and put $r+z=x$.

Then, by the nature of logarithms, the proposed equation $x^x=a$ will become

$$(r+z)\log.(r+z)=\log.a.$$

But $\log.(r+z)=\log.r(1+\frac{z}{r})=\log.r+\log.(1+\frac{z}{r})$

$=\log.r+\text{M}\{\frac{z}{r}-\frac{1}{2}(\frac{z}{r})^2+\frac{1}{3}(\frac{z}{r})^3-\text{\&c.}\}$; and, consequently,

$(r+z)\log.r+\text{M}(r+z)\{\frac{z}{r}-\frac{1}{2}(\frac{z}{r})^2+\frac{1}{3}(\frac{z}{r})^3-\text{\&c.}\}=\log.a.$

Whence, multiplying by $r+z$, and neglecting all the powers of z above the first, as being very small, we shall have $r\log.r+z\log.r+\text{M}z=\log.a$; or

$$z=\frac{\log.a-r\log.r}{\log.r+\text{M}},\text{ and }x=\frac{\log.a+\text{M}r}{\log.r+\text{M}}.$$

Which last formula will give a near approximate value of the unknown quantity sought.

And if the numeral value of x, in this expression, be put $=r'$, and $r'+z'=x$, we shall obtain, by a similar process,

$$z'=\frac{\log.a-r'\log.r'}{\log.r'+\text{M}},\text{ and }x=\frac{\log.a+\text{M}r'}{\log.r'+\text{M}},$$

which is a still nearer approximation; and thus, by proceeding in the same way, the value of x may be found to any degree of accuracy required.

BINOMIAL THEOREM.

(Y) The binomial theorem, as far as regards the generation of the coefficients of any integral powers from those of the preceding ones, was known to Vieta, Oughtred, and many of the more early writers on Arithmetic and Algebra, who formed tables for constructing them to any extent required; but Briggs, in his Trigonometria Britannica, printed in 1633, appears to have been the first who showed how the several terms of any integral power whatever, of a binomial, might be successively derived from each other, independently of any of the former powers, by a general rule; which was, in fact, giving, for this particular case, the substance of the theorem in words, but wanting the notation in symbols.

It may, however, be fairly questioned, as has been observed by Woodhouse, in his Principles of Analytical Calculation, whether Briggs knew how, even in the case of an integral exponent, to exhibit the law of the formation of the coefficients, under the form

$$\frac{m(m-1)(m-2)(m-3)\ldots[m-(n-1)]}{1\,.\,2\,.\,3\,.\,4\ldots n}$$

for though his method of determining the successive coefficients amounts to nearly the same thing, yet the advancement in analysis depended on the circumstance of the law which they observe being expressed in general terms, without which it could never have been extended to

those cases where the index is fractional or negative; so that Briggs, even in the case of integral powers, does not appear to be fully entitled to the invention of the binomial theorem, properly so called.

But, without insisting upon this circumstance, it is now universally agreed, that no one before Newton had ever thought of extracting roots by means of infinite series. He was the first who happily discovered, that by considering roots as powers, having fractional exponents, the same binomial series would be equally true for them all, whether the index should be fractional or integral, or the series finite or infinite; from which extension of the theorem, many important improvements, in the higher departments of science, have arisen, particularly in the construction of logarithms and the doctrine of series in general, which have since been carried to a great degree of perfection, and now form some of the most curious and interesting branches of analysis.

Newton, however, as is well known, left no demonstration of this celebrated theorem, but appears to have deduced it merely from an induction of particular cases; and though no doubt can be entertained of its truth, from its having been found to succeed in all the instances in which it has been applied, yet, agreeably to the rigour that ought to be observed in the establishment of every mathematical theory, and especially in a fundamental theorem, of such general use and application, it is necessary that as regular and strict a proof should

be given of it as the nature of the subject and the state of analysis will admit (*c*).

James Bernouilli, accordingly, undertook, soon after the theorem became known, to furnish a demonstration of this kind, which may be seen in his treatise entitled *Ars Conjectandi*, published in 1713; but his method of investigation is only applicable to the case of integral and affirmative powers, being nearly the same with that which was afterwards given by Stewart, in his Commentary on Newton's Quadrature of Curves. It is founded on the doctrine of combinations, and the properties of figurate numbers, which are there shown to involve in them the generation of the coefficients of a binomial; and, in the case before mentioned, where the index is a whole positive number, the theorem is clearly and satisfactorily explained.

Since that time, many attempts have been made to demonstrate the general case, or that in which the index of the binomial is either a whole number or a fraction; but most of these demonstrations, having been conducted either by the method of increments, the multinomial theorem of deMoivre, or by Fluxions, are commonly considered as imperfect and unsatisfactory; it being contrary to the principles of science, as well as to just reasoning, to employ, in a matter purely algebraical, notions and

(*c*) For an account of the means by which Newton was led to the discovery of the general theorem here mentioned, as well as for a more ample detail of all the most interesting particulars relating to this curious subject, see the article Binomial Theorem, which was given by the author, in the New Cyclopedia of Rees.

doctrines derived from other branches, or from an analysis, which is, in some measure, transcendental.

For these reasons, several eminent mathematicians, both in this country and on the continent, have endeavoured to investigate this formula on pure analytical principles, in a more natural and obvious way than had hitherto been done; one of the first of which attempts is that of Landen, in his Discourse concerning the Residual Analysis, published in 1764; and the next that of Epinus, in the eighth volume of the New Petersburg Memoirs. But the legitimacy of the former may be objected to, on account of its depending on vanishing fractions, and other considerations, of too intricate and abstract a nature, to be regarded as sufficiently convincing; and the latter, though very ingenious, is not less difficult and embarrassing; at least such is the opinion of Euler, who having himself first given a demonstration of this theorem, in which, like Maclaurin, he employed the differential calculus, or method of fluxions, was afterwards led to deduce it from the principles of Algebra alone; though he does not appear to have been much more successful than either of the former.

Lhuilier of Geneva, perceiving the defects and obscurity of these methods, gave a new demonstration of this formula, in one of the preliminary articles of his work, entitled *Principiorum calculi differentialis et integralis*, which is strictly elementary, and, abating for its length, and a fatiguing detail of particulars, which the nature of the subject does not seem to require, he appears to have accomplished his object; at least as far as the method

be adopted would allow; for it must be confessed, that neither this nor any other investigation, that had hitherto appeared, could be considered as possessing that degree of strictness and simplicity, which the purpose intended seems to require.

The cause of this failure, as the writer above mentioned has justly remarked, seems to be, that most mathematicians appear to have sought for some high origin of the theorem in question, distinct from the simple operations of multiplication, division, extraction of roots, &c.; and, instead of considering the nature of the operations it was known to comprehend, hoped to supercede them, by deductions drawn from abstruse and fine theories; whereas it is evident, that whatever imperfections these fundamental operations are attended with, the same must necessarily be attached to the binomial theorem; which being only a more general, but similar case, may likewise be said, in a certain sense, to be a method of trial and error; and, consequently, no demonstration of it can be expected to possess a greater degree of certainty than what is possessed by the leading operations themselves.

In order, therefore, to avoid entering into a too prolix investigation of the simple and well known elements, upon which the general formula depends, it will be sufficient to observe, that it is clearly manifest, from some of the first and most common rules of Algebra, that whatever may be the operation which the index (m) directs to be performed upon the expression $(a+x)^m$, whether of elevation, division, or extraction of roots, the terms of the resulting series will necessarily arise, by regular and

whole positive powers of x; and that the two first terms of this series will always be $a^m + ma^{m-1}x$; so that the entire expansion of it may be represented under the form

$$a^m + ma^{m-1}x + \text{A}a^{m-2}x^2 + \text{B}a^{m-3}x^3 + \text{C}a^{m-4}x^4 + \&c.$$

Where A, B, C, &c. are certain numeral coefficients, that are independent of the values of a and x; which last may be considered as denoting any quantities whatever. For, omitting the practical part of the process, which is taught by the above-mentioned rules, it will constantly be found, by performing the operations at length, in the usual way, that

$$(a+x)^2 = a^2 + 2ax + x^2$$
$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$$
$$(a+x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$$
&c. &c.

$$\frac{1}{a+x} = a^{-1} - a^{-2}x + a^{-3}x^2 - a^{-4}x^3 + \&c.$$
$$\frac{1}{(a+x)^2} = a^{-2} - 2a^{-3}x + 3a^{-4}x^2 - 4a^{-5}x^3 + \&c.$$
$$\frac{1}{(a+x)^3} = a^{-3} - 3a^{-4}x + 6a^{-6}x^2 - 10a^{-8}x^3 + \&c.$$
&c. &c.

$$(a+x)^{\frac{1}{2}} = a^{\frac{1}{2}} + \frac{1}{2}a^{-\frac{1}{2}}x - \frac{1}{8}a^{-\frac{3}{2}}x^2 + \frac{1}{16}a^{-\frac{5}{2}}x^3 - \&c.$$
$$(a+x)^{\frac{1}{3}} = a^{\frac{1}{3}} + \frac{1}{3}a^{-\frac{2}{3}}x - \frac{1}{9}a^{-\frac{5}{3}}x^2 + \frac{5}{81}a^{-\frac{8}{3}}x^3 - \&c.$$
$$(a+x)^{\frac{2}{3}} = a^{\frac{2}{3}} + \frac{2}{3}a^{-\frac{1}{3}}x - \frac{1}{9}a^{-\frac{4}{3}}x^2 + \frac{2}{81}a^{-\frac{8}{3}}x^3 - \&c.$$
&c. &c.

$$\frac{1}{(a+x)^{\frac{1}{4}}} = a^{-\frac{1}{4}} - \frac{1}{2}a^{-\frac{5}{4}}x + \frac{3}{8}a^{-\frac{9}{4}}x^2 - \frac{5}{16}a^{-\frac{13}{4}}x^3 + \&c.$$

$$\frac{1}{(a+x)^{\frac{1}{3}}} = a^{-\frac{1}{3}} - \frac{1}{3}a^{-\frac{4}{3}}x + \frac{2}{9}a^{-\frac{7}{3}}x^2 - \frac{7}{8}a^{-\frac{10}{3}}x^3 + \&c.$$

$$\frac{1}{(a+x)^{\frac{2}{3}}} = a^{-\frac{2}{3}} - \frac{2}{3}a^{-\frac{5}{3}}x + \frac{5}{9}a^{-\frac{8}{3}}x^2 - \frac{20}{81}a^{-\frac{11}{3}}x^3 + \&c.$$

&c. &c.

In all of which cases it is apparent, that the series proceeds according to the regular powers of x; and that the first term, in each of them, is the same as the power, or root, of the first term of the binomial to which it belongs; and, likewise, that the coefficient of x, in the second term, is always had by multiplying the index of the first term into that term, having its index diminished by 1.

And as these cases are of the same kind with those that are designed to be expressed in universal terms, by the general formula, it is in vain, as far as regards the first two terms of the expansion, to look for any other origin of them than what may be derived from these and other similar operations.

Hence, because $(a+x)^m = a^m(1+\frac{x}{a})^m$, if there be assumed $(a+x)^m = a^m + ma^{m-1}x + Ax^2 + Bx^3 + Cx^4$ &c.; or, which will render the process of investigation more simple, and equally answer the design proposed,

$$(1+\frac{x}{a})^m = 1 + A_1(\frac{x}{a}) + A_2(\frac{x}{a})^2 + A_3(\frac{x}{a})^3 + \&c. \quad . \quad . \quad . \quad (1)$$

it will only remain to determine the values of the coefficients A_1, A_2, A_3, &c. and to show the law of their

dependence on the index (m) of the operation by which they are produced.

For this purpose, let m denote any number whatever, whole or fractional, positive or negative; and for $\frac{x}{a}$, in the above formula, put $y+z$; then there will arise

$$(1+\frac{x}{a})^m = \{1+(y+z)\}^m = \{(1+y)+z\}^m.$$

Which being all identical expressions, their expansions, when taken according to the above form, will evidently be equal to each other.

Whence, as the numeral coefficients A_1, A_2, A_3, &c. of the developed formulæ, will not change for any value that may be given to a and x, provided the index (m) remains the same, the two latter may be exhibited under the forms

$$\{1+(y+z)\}^m =$$
$$1+A_1(y+z)+A_2(y+z)^2+A_3(y+z)^3\&c.$$

And

$$\{(1+y)+z\}^m =$$
$$(1+y)^m+A_1z(1+y)^{m-1}+A_2z^2(1+y)^{m-2}\&c.$$

And, consequently, by raising the several terms of the first of these series to their proper powers, and putting $1+y=p$ in the latter, we shall have

$$1+A_1(y+z)+A_2(y^2+2yz+z^2)+A_3(y^3+3y^2z+3yz^2+z^3)\&c. = p^m+A_1p^{m-1}z+A_2p^{m-2}z^2+A_3p^{m-3}z^3\&c.$$

Or, by ordering the terms so that those which are affected with the same power of z may be all brought together, and arranged under the same head, this last expression will stand thus:

$$\begin{array}{l|l|l|l}
1 + A_1 & z + A_2 & z^2 + A_3 & z^3 \text{ \&c.} \quad . \quad . \quad . \quad (2) \\
A_1 y + 2A_2 y & + 3A_3 y & + 4A_4 y & \\
A_2 y^2 + 3A_3 y^2 & + 6A_4 y^2 & + 10A_5 y^2 & \\
A_3 y^3 + 4A_4 y^3 & + 10A_5 y^3 & + 30A_6 y^3 & \\
\text{\&c.} & \text{\&c.} & \text{\&c.} &
\end{array}$$

$$= p^m + A_1 p^{m-1} z + A_2 p^{m-2} z^2 + A_3 p^{m-3} z^3 \text{\&c.}$$

In which equation it is evident, that both y and z are indeterminate, and independent of the values of A_1, A_2, A_3, &c. since the result here obtained arises solely from the substitution of these letters for x in equation (1).

Hence, as the coefficients, or multipliers of the like powers of z, are, in this case, identical, Art. (N) 16, we shall have, by comparing the first column of the left hand member with the first term of that on the right,

$$1 + A_1 y + A_2 y^2 + A_3 y^3 \text{\&c.} = p^m.$$

Which is an identity that verifies itself; since, by hypothesis, $p^m = (1 + y)^m$; and, according to the general formula,

$$(1 + y)^m = 1 + A_1 y + A_2 y^2 + A_3 y^3 + \text{\&c.}$$

Also, if the second of these columns be compared, in like manner, with the second term on the right, there will arise the new identity

$$A_1 + 2A_2 y + 3A_3 y^2 + 4A_4 y^3 + \text{\&c.} = A_1 p^{m-1}.$$

Which will be sufficient, independently of the rest of the terms, for determining the values of the coefficients A_1, A_2, A_3, &c. (*d*)

(*d*) Although we have here only employed those terms of the identity (2), which are affected with the first power of z, we have a right to conclude that all the rest of the corresponding terms of its two members are equally identical; for if this were not the

For since $A_1 p^{m-1} = A_1 \frac{p^m}{p} = \frac{A_1}{1+y}(1 + A_1 y + A_2 y^2 + A_3 y^3 \&c.)$, the equating this series with the last, and multiplying the left hand side by $1+y$, will give

$$\{A_1 + 2A_2 y + 3A_3 y^2 + 4a_4 y^3 \&c.\}(1+y) =$$
$$A_1 + A_1 A_1 y + A_1 A_2 y^2 + A_1 A_3 y^3 \&c.$$

And, therefore, by actually performing the operation, and arranging the terms accordingly, we shall have

$$\begin{array}{r|r|r|l} A_1 + 2A_2 & y + 3A_3 & y^2 + 4A_4 & y^3 + \&c. \\ + \quad A_1 & + 2A_2 & + 3A_3 & \end{array}$$
$$= A_1 + A_1 A_1 y + A_1 A_2 y^2 + A_1 A_3 y^3 + \&c.$$

From which last identity, there will obviously arise, by equating the homologous terms of its two members, the following relations of the coefficients:

$A_1 = A_1$	Or	$A_1 = A_1$
$2A_2 = A_1 A_1 - A_1$		$A_2 = \frac{A_1(A_1 - 1)}{2}$
$3A_3 = A_1 A_2 - 2A_2$		$A_3 = \frac{A_2(A_1 - 2)}{3}$
$4A_4 = A_1 A_3 - 3A_3$		$A_4 = \frac{A_3(A_1 - 3)}{4}$
. . . . .		. . . . .
$nA_n = A_1 A_{n-1} - (n-1)A_{n-1}$		$A_n = \frac{A_{n-1}[A_1 - (n-1)]}{n}$

And, consequently, as the coefficient A_1 of the second term of the expanded binomial, has been

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case there would arise several new equations, which, as the coefficients A_1, A_2, A_3, &c. have been already determined, it would be impossible to resolve; and, therefore, it would necessarily follow, that the development of $(1 + \frac{x}{a})^m$ could not be represented by $1 + A_1(\frac{x}{a}) + A_2(\frac{x}{a})^2 + A_3(\frac{x}{a})^3 \&c.$; which is contrary to the hypothesis.

shown to be equal, in all cases, to the index m, the last of these expressions will become of the form

$$A_1 = m$$

$$A_2 = \frac{m(m-1)}{2}$$

$$A_3 = \frac{m(m-1)(m-2)}{2 \,.\, 3}$$

$$A_4 = \frac{m(m-1)(m-2)(m-3)}{2 \,.\, 3 \,.\, 4}$$

$$\cdots\cdots\cdots\cdots$$

$$A_n = \frac{m(m-1)(m-2)(m-3) \,\ldots\ldots\, [m-(n-1)]}{2 \,.\, 3 \,.\, 4 \,.\, 5 \,\ldots\ldots\, n}$$

Where the law of the continuation of the terms, from A_4 to the general term A_n, is sufficiently evident.

Whence it follows, that, whether the index m be integral or fractional, positive or negative, the proposed binomial, $(a+x)^m$, when expanded, may always be exhibited under the form

$$a^m\left(1+\frac{x}{a}\right)^m =$$

$$a^m\left\{1 + m\left(\frac{x}{a}\right) + \frac{m(m-1)}{2}\left(\frac{x}{a}\right)^2 + \frac{m(m-1)(m-2)}{2 \,.\, 3}\left(\frac{x}{a}\right)^3 \&c.\right\}$$

Or

$$(a+x)^m =$$

$$a^m + ma^{m-1}x + \frac{m(m-1)}{2}a^{m-2}x^2 + \frac{m(m-1)(m-2)}{2 \,.\, 3}a^{m-3}x^3 \&c.$$

And if $-\frac{x}{a}$ be substituted in the place of $+\frac{x}{a}$, the same formula, will, in that case, be expressed as follows:

$$a^m(1-\frac{x}{a})^m =$$

$$a^m\{1-m(\frac{x}{a})+\frac{m(m-1)}{2}(\frac{x}{a})^2-\frac{m(m-1)(m-2)}{2\,.\,3}(\frac{x}{a})^3\&c.\}$$

Or

$$(a-x)^m =$$

$$a^m - ma^{m-1}x+\frac{m(m-1)}{2}a^{m-2}x^2-\frac{m(m-1)(m-2)}{2\,.\,3}a^{m-3}x^3\&c.$$

Where it is to be observed, that the series, in each of these cases, will terminate, when m is a whole positive number; but if m be negative, or fractional, it will run on *ad infinitum;* as neither of the factors $m-1$, $m-2$, $m-3$, &c. can, in these instances, become $=0$.

It may also be farther remarked, that when a and x, in these formulæ, are each equal to 1, we shall have, agreeably to such a substitution,

$$2^m = 1+m+$$

$$\frac{m(m-1)}{2}+\frac{m(m-1)(m-2)}{2\,.\,3}+\frac{m(m-1)(m-2)(m-3)}{2\,.\,3\,.\,4}+\&c.$$

And

$$0 = 1-m+$$

$$\frac{m(m-1)}{2}-\frac{m(m-1)(m-2)}{2\,.\,3}+\frac{m(m-1)(m-2)(m-3)}{2\,.\,3\,.\,4}-\&c.$$

From which it appears, that the sum of the coefficients, arising out of the development of the mth power, or root, of any binomial, is equal to 2^m; and that the sum of the coefficients of the odd terms of the mth power, or root, of a residual quantity, is equal to the sum of the coefficients of the even terms.

It may here also be observed, that the several forms of this theorem, given in the note to the practical rule, Part I, may be readily determined by substituting

$$\frac{a}{1-\frac{x}{a+x}} \quad \text{or} \quad \frac{2a}{1+\frac{a-x}{a+x}}$$

in the place of their equal $a+x$, and expanding the expression according to the first, or usual form (*e*).

MULTIPLICATION AND DIVISION OF SERIES.

(z) THE article respecting the multiplication and division of series, to which this refers, in the former part of the work, requires no farther notice than barely to observe, that the results obtained by means of the peculiar mode of notation there used, render the law of the coefficients far more simple and perspicuous than it would have appeared if the operations had been performed according to the method commonly employed in these cases.

OF THE MULTINOMIAL THEOREM.

(A) P. 206. THE multinomial theorem may be readily demonstrated, by putting all the terms after the

(*e*) The demonstration above given, (which it is presumed will be found as simple and satisfactory as the nature of the subject will admit,) is founded upon a similar principle with that first laid down by Lagrange in his *Théorie des Fonctions Analytiques*, to which work the reader is referred for farther information on this subject, as well as for whatever relates to the doctrine of expanded functions in general; which is there treated of in a way worthy the genius of the author.

first equal to some new quantity, and then treating it as a binomial; but the process will be rendered considerably more simple by assuming

$$(A_0 + A_1x + A_2x^2 + A_3x^3 \&c.)^m =$$
$$B_0 + B_1x + B_2x^2 + B_3x^3 \&c.;$$

which, from what has been shown respecting the expansion of a binomial, Art. (Y), may always be legally done.

For if the first member of this equation be put under the form $(A_0 + u)^m$, the series arising from its expansion will ascend, according to the article last mentioned, by the successive positive powers of u.

And, consequently, if instead of u, in this series, there be put its former value, the substitution so made will introduce only the whole positive powers of x.

Hence, supposing x to be increased by z, or $x+z$ to be put in the place of x, we shall have

$$\{A_0 + A_1(x+z) + A_2(x+z)^2 + A_3(x+z)^3 \&c.\}^m =$$
$$B_0 + B_1(x+z) + B_2(x+z)^2 + B_3(x+z)^3 \&c. \ldots\ldots (1)$$

In which last equation, if each of the terms of its two members be developed in the usual manner, the terms that multiply the like powers of x and z will be identical.

Let, therefore, the several terms of the first of these members, when squared, cubed, &c. be arranged according to the powers of z, and it will become of the form

$$\{A_0 + A_1x + A_2x^2 + A_3x^3 \&c. +$$
$$(A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 \&c.)z +$$
$$(A_2 + 3A_3x + 6A_4x^2 \&c.)z^2 \&c.\}^m.$$

Or, if there be put, for the sake of abridging the operation,

$$A_0 + A_1x + A_2x^2 + A_3x^3 \&c. = P,$$
$$A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 \&c. = Q,$$
$$A_2 + 3A_3x + 6A_4x^2 + \&c. = R, \&c.$$

the same expression may be exhibited as follows:

$$(P + Qz + Rz^2 + Sz^3 + Tz^4 \&c.)^m.$$

Which last formula, by considering all the terms after the first as one, and treating it as a binomial, may be represented by

$$P^m + mP^{m-1}(Qz + Rz^2 + Sz^3 \&c.) +$$
$$\frac{m(m-1)}{2}P^{m-2}(Qz + Rz^2 + Sz^3 \&c.)^2 \&c. \ldots (2)$$

Also, the second member of the same equation, when its several terms are raised to their proper powers, and arranged according to the powers of z, will become of the form

$$B_0 + B_1x + B_2x^2 + B_3x^3 \&c. +$$
$$(B_1 + 2B_2x + 3B_3x_2 + 4B_4x^3 \&c.)z +$$
$$(B_2 + 3B_3x + 6B_4x^2 \&c.)z^2 + \ldots (3)$$

Hence, since the formulæ (2) and (3) are the two members of equation (1) under another form, we shall have, by comparing the first term of one of them with that of the other,

$$B_0 + B_1x + B_2x^2 + B_3x^3 + B_4x^4 \&c. = P^m;$$

which is an identity that verifies itself; since, by hypothesis, and the above substitution,

$$(A_0 + A_1x + A_2x^2 + A_3x^3 \&c.)^m, \text{ or}$$
$$P^m = B_0 + B_1x + B_2x^2 + B_3x^3 \&c.$$

In like manner, taking such of the terms only, in each of these formulæ, as involve the first, or

simple power of z, and comparing them as above, there will arise the new identity

$$B_1 + 2B_2x + 3B_3x^2 + 4B_4x^4 \&c. = m\mathrm{Q}\mathrm{P}^{m-1}.$$

Which, as in the case of the bionmial, will be sufficient, independently of the rest of the terms, for determining the value of the coefficients

$$B_1,\ B_2,\ B_3,\ \&c.$$

For, since $m\mathrm{Q}\mathrm{P}^{m-1} = \frac{m\mathrm{Q}\mathrm{P}^m}{\mathrm{P}}$, we shall have, by substituting for Q, P^m and P, their values, as above given, and multiplying the left hand member of the identity by the denominator of the right hand member,

$$\begin{gathered}(B_1 + 2B_2x + 3B_3x^2 + 4B_4x^3 \&c.)\\ (A_0 + A_1x + A_2x^2 + A_3x^3 \&c.) = \\ m(A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 \&c.)\\ (B_0 + B_1x + B_2x^2 + B^3x^3 \&c.).\end{gathered}$$

And, consequently, by actually performing the multiplications here indicated, there will arise

$$\begin{array}{r|r|r|l}
A_0B_1 + 2A_0B_2 & x + 3A_0B_3 & x^2 + 4A_0B_4 & x^3 \&c. \\
+ A_1B_1 & 2A_1B_2 & + 3A_1B_3 & \\
 & + A_2B_1 & + 2A_2B_2 & \\
 & & + A_3B_1 &
\end{array}$$

$$\begin{array}{r|r|r|l}
= mA_1B_0 + mA_1B_1 & x + mA_1B_2 & x^2 + mA_1B_3 & x^3 \&c. \\
+ 2mA_2B_0 & + 2mA_2B_1 & + 2mA_2B_2 & \\
 & + 3mA_3B_0 & + 3mA_3B_1 & \\
 & & + 4mA_4B_0 &
\end{array}$$

Whence, by comparing the corresponding coefficients of the homologous powers of x, in the two members of this last equation, we shall have

$$A_0B_1 = mA_1B_0$$
$$2A_0B_2 = (m-1)A_1B_1 + 2mA_2B_0$$
$$3A_0B_3 = (m-2)A_1B_2 + (2m-1)A_2B_1 + 3mA_3B_0$$
$$4A_0B_4 = (m-3)A_1B_3 + (2m-2)A_2B_2 + (3m-1)A_3B_1 + 4mA_4B_0$$
$$5A_0B_5 = (m-4)A_1B_4 + (2m-3)A_2B_3 + (3m-2)A_3B_2 + (4m-1)A_4B_1 + 5mA_5B_0$$
&c. &c.

Where the law by which the several terms are derived is sufficiently evident; and, consequently, if A_0 be put for B_0, which is its value, when x in the original series is $=0$, the coefficients B_1, B_2, B_3, &c. will be determined (*f*).

OF THE REVERSION OF SERIES.

(B) P. 208. THE rule given in the former part of the work, for reverting a series of the form

$$ax + bx^2 + cx^3 + dx^4 + ex^5 \&c. = y,$$

or finding the value of x in terms of y, may be demonstrated as follows; assume

$$x = Ay + By^2 + Cy^3 + Dy^4 + Ey^5 \&c.$$

Then, if this value be substituted for x and its powers, in the above equation, we shall have

$$ax + bx^2 + cx^3 + dx^4 + ex^5 \&c. \text{ or}$$

(*f*) It may here be observed, that any algebraical function whatever may be developed by some of the preceding methods; for the most general expression of this kind, that can be proposed, must consist of two or more combinations of monomials or polynomials, elevated to some positive or negative powers, which are either integral or fractional.

$$y = \left| \begin{array}{l} aAy + aB \\ \quad + bA^2 \end{array} \right| \begin{array}{l} y^2 + aC \\ \quad + 2bAB \\ \quad + cA^3 \end{array} \left| \begin{array}{l} y^3 + aD \\ \quad + 2bAC \\ \quad + bB^2 \\ \quad + 3cA^2B \\ \quad + dA^4 \end{array} \right| \begin{array}{l} y^4 + aE \\ \quad + 2bAD \\ \quad + 2bBC \\ \quad + 3cA^2C \\ \quad + 3cAB^2 \\ \quad + 4dA^3B \\ \quad + eA^5 \end{array} \left| y^5 \&c. \right.$$

And, consequently, if the homologous terms be equated, there will arise the following results:

$$aA = 1$$
$$aB + bA^2 = 0$$
$$aC + 2bAB + cA^3 = 0$$
$$aD + 2bAC + bB^2 + 3cA^2B + dA^4 = 0$$
$$aE + 2bAD + 2bBC + 3cA^2C + 3cAB^2 + 4dA^3B + eA^5 = 0$$
$$\&c. \qquad\qquad \&c.$$

Where, finding the value of A, in the first of these equations, that of B in the second, and so on; and then substituting them, as they arise, in the succeeding equations, we shall readily obtain

$$A = \frac{1}{a}$$
$$B = -\frac{b}{a^2}$$
$$C = \frac{2b^2 - ac}{a^5}$$
$$D = -\frac{5b^3 - 5abc + a^2d}{a^7}$$
$$E = \frac{14b^4 - 21ab^2c + 3a^2c^2 + 6a^2bd - a^3e}{a^9}$$

Which are the coefficients of y and its powers, in the inverted equation, agreeably to the rule (g).

(g) If the series that is to be inverted be of the form $a + ax + bx^2 + cx^3 + dx^4 \&c. = y$,

2. When the series consists of the odd powers of x, or is of the form

$$ax + bx^3 + cx^5 + dx^7 + ex^9 \&c. = y,$$

the value of x in terms of y, may be obtained in a similar manner, by assuming

$$x = \mathrm{A}y + \mathrm{B}y^3 + \mathrm{C}y^5 + \mathrm{D}y^7 + \mathrm{E}y^9 \&c.$$

For if this value be substituted for x and its powers, as in the former case, we shall have

$$ax + bx^3 + cx^5 + dx^7 + ex^9 \&c. \text{ or}$$

$$y = \left| \begin{array}{l|l|l|l|l} a\mathrm{A}y + a\mathrm{B} & y^3 + a\mathrm{C} & y^5 + a\mathrm{D} & y^7 + a\mathrm{E} & y^9 \&c. \\ \quad + b\mathrm{A}^3 & \quad + 3b\mathrm{A}^2\mathrm{B} & \quad + 3b\mathrm{A}^2\mathrm{C} & \quad + 3b\mathrm{A}^2\mathrm{D} & \\ & \quad + c\mathrm{A}^5 & \quad + 3b\mathrm{A}\mathrm{B}^2 & \quad + 6b\mathrm{A}\mathrm{B}\mathrm{C} & \\ & & \quad + 5c\mathrm{A}^4\mathrm{B} & \quad + b\mathrm{B}^3 & \\ & & \quad + d\mathrm{A}^7 & \quad + 5c\mathrm{A}^4\mathrm{C} & \\ & & & \quad + 10c\mathrm{A}^3\mathrm{B}^2 & \\ & & & \quad + 7d\mathrm{A}^6\mathrm{B} & \\ & & & \quad + e\mathrm{A}^9 & \end{array} \right|$$

And, consequently, by comparing the homologous terms, as before, there will arise the following results:

$a\mathrm{A} = 1$

$a\mathrm{B} + b\mathrm{A}^3 = 0$

$a\mathrm{C} + 3b\mathrm{A}^2\mathrm{B} + c\mathrm{A}^5 = 0$

$a\mathrm{D} + 3b\mathrm{A}^2\mathrm{C} + 3b\mathrm{A}\mathrm{B}^2 + 5c\mathrm{A}^4\mathrm{B} + d\mathrm{A}^7 = 0$

$a\mathrm{E} + 3b\mathrm{A}^2\mathrm{D} + 6b\mathrm{A}\mathrm{B}\mathrm{C} + b\mathrm{B}^3 + 5c\mathrm{A}^4\mathrm{C} + 10c\mathrm{A}^3\mathrm{B}^2 + 7d\mathrm{A}^6\mathrm{B} + e\mathrm{A}^9 = 0$

&c. &c.

Where, finding the value of A in the first of these equations, that of B in the second; and so

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a must be taken to the other side of the equation, and $y - a$ put $= z$, when the coefficients will be the same for z, as they were before for y.

on; and then substituting them, as they arise, in the succeeding equations, we shall readily obtain

$$\mathrm{A} = \frac{1}{a}$$

$$\mathrm{B} = -\frac{b}{a^4}$$

$$\mathrm{C} = \frac{3b^2 - ac}{a^7}$$

$$\mathrm{D} = -\frac{12b^3 + a^2d - 8abc}{a^{10}}$$

$$\mathrm{E} = \frac{55b^4 - 55ab^2c + 10a^2bd + 5a^2c^2 - a^3e}{a^{13}}$$

&c. &c.

Which are the coefficients of y and its powers, in the reverted equation, for this case of the rule.

3. When there are two series, consisting of the successive powers of x and y, as

$$ax + bx^2 + cx^3 + dx^4 \&c. = \alpha y + \beta y^2 + \gamma y^3 + \delta y^4 \&c.$$

Let there be assumed, as in the instances before given,

$$x = \mathrm{A}y + \mathrm{B}y^2 + \mathrm{C}y^3 + \mathrm{D}y^4 + \mathrm{E}y^5 \&c.$$

Then, if this value be substituted for x and its powers, in the left hand member of the proposed equation, we shall have

$$\alpha y + \beta y^2 + \gamma y^3 + \delta y^4 + \epsilon y^5 \&c. =$$

$a\mathrm{A}y + a\mathrm{B}$	$y^2 + a\mathrm{C}$	$y^3 + a\mathrm{D}$	$y^4 + a\mathrm{E}$	$y^5 \&c.$
$+ b\mathrm{A}^2$	$+ 2b\mathrm{AB}$	$+ 2b\mathrm{AC}$	$+ 2b\mathrm{AD}$	
	$+ c\mathrm{A}^3$	$+ b\mathrm{B}^2$	$+ 2b\mathrm{BC}$	
		$+ 3c\mathrm{A}^2\mathrm{B}$	$+ 3c\mathrm{A}^2\mathrm{C}$	
		$+ d\mathrm{A}^4$	$+ 3c\mathrm{AB}^2$	
			$+ 4d\mathrm{A}^3\mathrm{B}$	
			$+ e\mathrm{A}^5$	

And, consequently, by equating the homologous terms, there will arise the following results:

$$aA = \alpha$$
$$aB + bA^2 = \beta$$
$$aC + 2bAB + cA^3 = \gamma$$
$$aD + 2bAC + bB^2 + 3cA^2B + dA^4 = \delta$$
$$aE + 2bAD + 2bBC + 3cA^2C + 3cAB^2 + 4dA^3B + eA^5 = \varepsilon$$

Where, finding the values of A, B, C &c. as in the former cases, and then substituting them in the succeeding equations, we shall readily obtain

$$A = \frac{\alpha}{a}$$
$$B = \frac{\beta - bA^2}{a}$$
$$C = \frac{\gamma - 2bAB - cA^3}{a}$$
$$D = \frac{\delta - bB^2 - 2bAC - 3cA^2B - dA^4}{a}$$
$$E = \frac{\varepsilon - 2bBC - 2bAD - 3cAB^2 - 3cA^2C - 4dA^3B - eA^5}{a}$$

Which are the coefficients of the terms of the series that gives the value of x in terms of y.

OF THE METHOD OF INDETERMINATE COEFFICIENTS.

(c) P. 214. THE method of indeterminate coefficients, which has been employed in several of the preceding articles, having been sufficiently explained in the former part of the work, it will here only be necessary to observe, that every algebraic function, containing one unknown quantity, or that quantity and its powers, may be developed into a series of the form

$$A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 \text{ \&c.}$$

of which the terms involve only the whole positive

powers of the unknown quantity x, and certain coefficients A_0, A_1, A_2, &c. which are wholly independent of the value of that quantity.

This is verified by the expansions of all the functions of this kind yet known; and the particular series that is to be adopted, in each case, will be readily determined, by putting $x=0$ and observing the nature of the result.

Thus, if the function to be developed is to become equal to some particular quantity for $x=0$, the first term of the series must be taken singly, or without an x; and if it is to become equal to zero, for $x=0$, the first term must contain x.

Also, if any term be either omitted, or introduced into the series, that does not belong to it, the comparisons of the homologous terms will always show the impropriety of it, by their leading to absurd equations.

Thus, taking, for instance, as a partial example of the first of these cases,

$$\frac{1}{1+x}=A_0+A_1x+A_2x^2+A_3x^3 \text{ \&c.}$$

it is plain, that when $x=0$, we shall have $\frac{1}{1}=A_0$, or $A_0=1$; which is, therefore, the first term of the series.

In like manner, if we take a fraction of the following kind,

$$\frac{x}{1+x}=A_1x+A_2x^2+A_3x^3+A_4x^4 \text{ \&c.}$$

it is equally evident, that the first term of the series must contain x, for $x=0$, or otherwise $\frac{0}{1}$ would be

equal to some determinate quantity, which is impossible.

Also, if there be taken, as an example of the last of the above cases,

$$\frac{1}{1-x} = A_0 + A_1 x + \ldots + A_3 x^3 + A_4 x^4 \text{ \&c.}$$

where the third term $A_2 x^2$ of the series is omitted, it would be found, by multiplying the right hand member of the identity by $1-x$, and equating the homologous coefficients, that $A_0 = 1$, $A_1 = 1$, and $A_3 = 1$, which is absurd.

Again, let there be taken, as an additional example, the well known formula

$$\frac{x^m - v^m}{x-v} = x^{m-1} + A_2 x^{m-2} + A_3 x^{m-3} + \ldots + v^{m-1} (h);$$

where m being a whole positive number, it readily appears that the series is rightly assumed; because, for $x=0$, the development is reduced to v^{m-1}, and for $v=0$ it is reduced to x^{m-1}.

Hence, multiplying the right hand member by $x-v$, we shall have

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(*h*) In the case where the index is a fractional number, which requires a different investigation, the same formula may be exhibited under the form

$$\frac{x^{\frac{m}{n}} - v^{\frac{m}{n}}}{x-v} = x^{\frac{m}{n}-1} \left| \frac{1 + \left(\frac{x}{v}\right) + \left(\frac{x}{v}\right)^2 + \left(\frac{x}{v}\right)^3 + \ldots + \left(\frac{x}{v}\right)^{m-1}}{1 + \left(\frac{x}{v}\right)^{\frac{m}{n}} + \left(\frac{x}{v}\right)^{\frac{2m}{n}} + \left(\frac{x}{v}\right)^{\frac{3m}{n}} + \ldots + \left(\frac{x}{v}\right)^{\frac{(m-1)m}{n}}} \right.$$

being, in this state, a similar theorem to that laid down by Landen, in his Residual Analysis, from which he attempted to derive the doctrine of fluxions.

$$x^m - v^m = x^m + A_2 \left| \begin{matrix} x^{m-1} + A_3 \\ -v \end{matrix} \right| \begin{matrix} x^{m-2} + A_4 \\ -A_2 v \end{matrix} \left| \begin{matrix} x^{m-3} \ldots + v^{m-1} x \\ -A_3 v \end{matrix} \right| \begin{matrix} \ldots - v^m \\ \ldots \end{matrix}$$

And, consequently, by taking the coefficients of the several terms of the product equal to 0, there will arise the following relations:

$$\begin{matrix} A_2 - v = 0 \\ A_3 - A_2 v = 0 \\ A_4 - A_3 v = 0 \\ A_5 - A_4 v = 0 \\ \&c. \end{matrix} \quad \Bigg| \quad \text{Or} \quad \Bigg| \quad \begin{matrix} A_2 = v \\ A_3 = v^2 \\ A_4 = v^3 \\ A_5 = v^4 \\ \&c. \end{matrix}$$

From which it is evident, that the proposed formula, when expanded, will become

$$\frac{x^m - v^m}{x - v} = x^{m-1} + vx^{m-2} + v^2 x^{m-3} + v^3 x^{m-4} + \ldots + v^{m-1}.$$

In like manner, supposing, when m is a whole number, the following polynomial, or series,

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \ldots + Tx + V,$$

to be exactly divisible by $x - a$, we shall have, by assuming a certain series for the quotient Q,

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \ldots + Tx + V =$$
$$(x - a)(x^{m-1} + A'x^{m-2} + B'x^{m-3} + \ldots + S'x + T');$$

where A, B, C, &c. are supposed to be known coefficients, and A′, B′, C′, &c. other coefficients that are to be determined.

Then, by actually performing the multiplication indicated in the second member of the identity, there will arise,

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \ldots + Tx + V$$
$$= \left| \begin{matrix} x^m + A' \\ -a \end{matrix} \right| \begin{matrix} x^{m-1} + B' \\ -aA' \end{matrix} \left| \begin{matrix} x^{m-2} + C' \\ -aB' \end{matrix} \right| \begin{matrix} x^{m-3} + S' \\ -aR' \end{matrix} \left| \begin{matrix} x^2 + T' \\ -aS' \end{matrix} \right| \begin{matrix} x - aT' \\ \end{matrix}$$

In which case, as the corresponding powers of x

in both series are similar, that for the development of the quotient appears to have been rightly assumed.

Wherefore, bringing all the terms to the right hand side of the equation, we shall have

$$\begin{array}{r|l|l|l

So that, from these instances, and others of a similar kind, that might be given, the form of the series for any expanded algebraical function, and the values of the assumed coefficients, may be readily found.

OF VANISHING FRACTIONS.

(D) P. 217. THE notions commonly received, at present, respecting the nature and use of vanishing fractions, or such as become of the form $\frac{0}{0}$, when some particular value is given to the unknown quantity, appear to have arisen, as before remarked, from certain peculiarities attending the application of the fluxional analysis to the doctrine of curves; in some cases of which it was sufficiently evident, from the properties of the figure under consideration, that this symbol, though apparently nugatory, had a real fixed value; and in others that it might be 0, finite, or infinite, according to the nature of the proposition from which it was derived.

Hence, as the known values of the expression, in these instances, were found to agree with those which arise from applying the hypothesis in question to the reduced fraction, according to the methods that have been laid down and elucidated in the practical article, Part 1, it seemed reasonable to conclude that these rules should be made general for all similar cases, in whatever way they might arise; or otherwise an anomalous principle must have been adopted upon this occasion, which would have rendered the algorithm of the science

uncertain, and the results of calculation at variance with each other.

It must be confessed, however, that any direct investigation of this subject is attended with considerable difficulties, owing to our being unable, in certain cases, to ascertain, in a clear and satisfactory manner, what effect 0, as an algebraical symbol of a restricted signification, will produce, when mixed, or connected with, other characters which are taken generally, or without any regard to their numeral values; so that the truth of the mode of solution, here pointed out, must be left to rest, in a great measure, upon the evidence derived from analogical considerations, similar to those above stated.

With respect to the other symbols, that are mentioned in the article referred to, it is plain, that 0 added to or subtracted from any quantity, makes it neither greater nor less; that is,

$$a+0=a, \text{ and } a-0=a.$$

Also, if nought be multiplied or divided by any quantity, both the product and quotient will be nought; because any number of times 0, or any part of 0, is 0: that is,

$$0\times a, \text{ or } a\times 0=0, \text{ and } \frac{0}{a}=0.$$

From this property, it likewise follows, that nought divided by nought, is a finite quantity, of some kind or other.

For since $0\times a=0$, or $0=0\times a$, it is evident, that $\frac{0}{0}=a$.

Farther, if any finite quantity be divided by 0, the quotient will be infinite.

For let $\frac{b}{a}=q$, then, if b remains the same, it is plain, the less a is, the greater will be the quotient q; whence, if a be indefinitely small, q will be indefinitely great; and, consequently, when a is 0, the quotient q will be infinite: that is,

$$\frac{b}{0}, \text{ or } \frac{1}{0}=\infty.$$

OF FIGURATE AND POLYGONAL NUMBERS.

(E) P. 223. THE whole theory of figurate numbers, as has been observed in the note to article (E), Part I, is founded upon the following proposition, which is equivalent to the one there mentioned,

The $n+1$th term of any order of figurate numbers, is equal to the sum of the nth term of that order and the $n+1$th term of the next inferior order; and the nth term of any order is equal to the sum of the n first terms of the preceding order.

For the purpose of demonstrating which, let there be taken, as in the article above referred to, the following series of these numbers.

Order.	*Figurate Numbers.*	*General Terms.*
1st.	1, 2, 3, 4, 5 &c.	n
2d.	1, 3, 6, 10, 15 &c.	$\frac{n(n+1)}{1.2}$
3d.	1, 4, 10, 20, 35 &c.	$\frac{n(n+1)(n+2)}{1.2.3}$
4th.	1, 5, 15, 35, 70 &c.	$\frac{n(n+1)(n+2)(n+3)}{1.2.3.4}$

nth order, $\frac{n(n+1)(n+2)(n+3)\ \ldots\ [n+(m-1)]}{1\,.\,2\,.\,3\,.\,4\ \ldots\ m}$

Then it is plain, that $n+1$, which is the $n+1$th

term of the 1st order, being added to $\frac{n(n+1)}{1.2}$, which is the nth term of the 2d order, gives

$$\frac{n(n+1)}{1.2}+n+1=\frac{(n+1)(n+2)}{1.2}$$

which is the $n+1$th term of the 2d order.

Also, $\frac{(n+1)(n+2)}{1.2}$, which is the $n+1$th term of the 2d order, being added to $\frac{n(n+1)(n+2)}{1.2.3}$, which is the nth term of the 3d order, gives

$$\frac{(n+1)(n+2)(n+3)}{1.2.3}$$

which is the $n+1$th term of the 3d order.

And generally

$$\frac{n(n+1)(n+2)(n+3)\ \ldots\ldots\ [n(m-2)]}{1.2.3\ \ldots\ldots\ (m-1)}$$

which is the $n+1$th term of the $m-1$th order, added to

$$\frac{n(n+1)(n+2)(n+3)\ \ldots\ldots\ [n+(m-1)]}{1.2.3\ \ldots\ldots\ m}$$

which is the nth term of the mth order, gives

$$\frac{n(n+1)(n+2)(n+3)\ \ldots\ldots\ (n+m)}{1.2.3\ \ldots\ldots\ m+1}$$

which is the $n+1$th term of the mth order.

From which, the general law of the formation of the terms is sufficiently manifest.

And since the second term of any order is equal to the sum of the two first terms of the preceding order, the third term will be equal to the sum of the three first terms of the preceding order; and so on (*i*).

(*i*) The method here observed, in treating of this proposition, is similar to that followed by Legendre, in his *Essai sur*

Whence, generally, the nth term of any order is equal to the first n terms of the next inferior order.

Again, if the nth term of the first order, or of the natural series, 1, 2, 3, 4, 5 &c. be multiplied by the $n+1$th term of any order m, the product will be equal to $m+2$ times the nth term of the $m+1$th order.

Which last proposition is the celebrated theorem of Fermat, alluded to in the corresponding article, Part I, the truth of which may be readily shown form the formulæ above given: thus,

$$n(n+1) = 2 \times \frac{n(n+1)}{1 \,.\, 2}$$

$$\frac{n(n+1)(n+2)}{1 \,.\, 2} = 3 \times \frac{n(n+1)(n+2)}{1 \,.\, 2 \,.\, 3}$$

$$\frac{n(n+1)(n+2)(n+3)}{1 \,.\, 2 \,.\, 3} = 4 \times \frac{n(n+1)(n+2)(n+3)}{1 \,.\, 2 \,.\, 3 \,.\, 4}$$

and so on for any other order whatever.

And in a manner equally easy, it may be shown, that every polygonal number, of the denomination m, is represented by the general formula

$$\frac{(m-2)n^2 - (m-4)n}{2}.$$

For, by Art. (E), Part I, p. 223, every $m$gonal number is the sum of n terms of a series of arithmetical proportionals, beginning with 1, and having the common difference $m-2$.

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la Théorie des Nombres, who, by deducing the law of the generation of the terms from the forms of the series, instead of ascertaining the forms of the series from the law of their generation, has rendered the subject much more simple and perspicuous, than by adhering to the manner in which it had been usually investigated.

Whence, putting $m-2=d$, we shall have

$$1+(1+d)+(1+2d)+(1+3d)\ldots\{1+(n-1)d\};$$

which, by the common rule for the summation of such a series, is equal to

$$\frac{n}{2}\{2+(n-1)d\}.$$

And, if $m-2$ be now substituted for d, this last expression will become

$$\frac{(m-2)n^2-(m-4)n}{2};$$

which is the same as the formula given above.

Also, when any polygonal number and its denomination is given, if there be put

$$\frac{(m-2)n^2-(m-4)n}{2}=p,$$

we shall readily obtain, by the solution of a quadratic equation,

$$n=\frac{m-4\pm\sqrt{[2p(m-2)+(m-4)^2]}}{2m-4};$$

which is a general formula for the square root of any polygonal number.

OF THE INDETERMINATE ANALYSIS.

(F) P. 227. HAVING sufficiently explained, in the corresponding Article of the practical part of the present work, whatever relates to the solution of indeterminate equations of the first degree, it will here only be necessary to show under what conditions questions of this kind are resolvable, and in what cases they are impossible.

For which purpose let there be premised the following *lemma:*

1. If a and b be any two numbers prime to each other, and the terms of the series

$$b,\ 2b,\ 3b,\ 4b,\ \ldots\ldots\ (a-1)b,$$

be respectively divided by a, they will each leave a different remainder.

For, if not, let any two of them, as mb, nb, when divided by a, give the quotients q, q', and leave the same remainder r.

Then, since $mb=qa+r$, and $nb=q'a+r$, it is evident, that each side of the equation

$$mb-nb=qa-q'a,$$

will be divisible by a, because the right hand member is so.

But this is impossible, since the factor b, in the left hand member,

$$mb-nb,\ \text{or}\ b(m-n),$$

is prime to a, and $m-n=a$, because, by the proposition, both m and n are so.

Whence mb cannot be $=qa+r$, and $nb=q'a+r$; and, consequently, no two of these terms can leave the same remainder, when divided by a.

Cor. 1. Since, therefore, all the remainders, which are $a-1$ in number, arising from the division of the several terms of the series

$$b,\ 2b,\ 3b,\ 4b,\ \ldots\ldots\ (a-1)b$$

by a, are different from each other, and each of them necessarily less than a, it follows, that these remainders will include all the numbers from 1 to $a-1$ inclusively.

And in the same manner it may be showu, that if the terms of the series

$$a,\ 2a,\ 3a,\ 4a,\ \ldots\ldots\ (b-1)a,$$

be respectively divided by b, they will leave different remainders, each less than b, which include all the numbers from 1 to $b-1$.

Cor. 2. Hence, since some one of the terms, in each of these series, will leave a remainder 1, a number $y' = a$, or a number $x' = b$, may be found, when a and b are prime to each other, that will render $by'-1$ divisible by a, or $ax'-1$ divisible by b.

From which principle it is also farther obvious, that the equation

$$ax' - by' = \pm 1$$

is always possible in whole numbers; since $y' = \frac{ax'-1}{b} =$ an integer, and $x' = \frac{by'-1}{a} =$ an integer.

2. If a and b be prime to each other, the equation

$$ax - by = \pm c$$

is always possible, and will admit of an infinite number of answers in whole numbers.

For since the equation $ax' - by' = \pm 1$ has been shown to be possible, let each of its sides be multiplied by c, and there will arise

$$acx' - bcy' = \pm c.$$

Whence, by putting $cx' = x$, and $cy' = y$, we shall have

$$ax - by = \pm c;$$

which is, therefore, also possible, when a is prime to b.

But if a and b have a common measure, which is not a divisor of c, the equation is impossible; as has been shown in the note to the practical rule.

Again, if one answer to the proposed equation,

$$ax - by = \pm c,$$

be known, an indefinite number of others may be found, as follows:

Let the two values, $x=p$ and $y=q$, be such as will satisfy the conditions of the question; then we shall have

$$ap-bq=\pm c.$$

And, consequently, by comparing this equation with the first

$$ax-by=ap-bq; \text{ or}$$
$$\frac{x-p}{y-q}=\frac{b}{a}=\frac{mb}{ma}.$$

Where b being prime to a, $\frac{b}{a}$ is a fraction in its lowest terms; whence if m be made to represent any indefinite number, it follows, that

$$x-p=mb \text{ and } y-q=ma.$$

And, therefore, by transposition, there will arise

$$x=p+mb \text{ and } y=q+ma.$$

In which expressions, if 0, 1, 2, 3, &c. be successively substituted for the indefinite number m, we shall have

$$x=p=p+b=p+2b=p+3b \ldots\ldots =p+mb,$$
$$y=q=q+a=q+2a=q+3a \ldots\ldots =q+ma.$$

Each of which quantities will satisfy the conditions of the question (*k*).

(*k*) The first solution of this problem is due to *M. Bachet de Meziriac*, the editor of Diophantus, who gave it in the second edition of his mathematical recreations, entitled *Problemes plaisans et delectables*, &c. printed in 1612; but the question is here barely announced, being only to be found complete in the edition of 1624.

Cor. It is evident, from the above expressions, that p and q being whole numbers, m may always be so taken, that x, in the equation

$$ax - by = \pm c,$$

shall be less than b, or y less than a.

3. If a, b, c, be all prime to each other, the equation

$$ax + by = c$$

is always possible, when $c = \{ab - (a + b)\}$.

For let $c = (ab - a - b) + r$; then, since the equation, in this case, becomes

$$ax + by = (ab - a - b) + r,$$

the possibility of it will evidently depend upon

$$x = \frac{ab - a - b - by + r}{a}$$

being some whole or integer number.

But this equation, when its several terms are divided by a, is the same as

$$x = b - 1 - \frac{(y + 1)b - r}{a};$$

and, therefore, in this state of the proposition, it depends upon the possibility of

$$\frac{(y + 1)b - r}{a} = x' \text{ being an integer.}$$

Or, by putting $y + 1 = y'$, upon the possibility of the equation

$$by' - ax' = r.$$

But, by the corollary to the last proposition, this may be always established, so that $y' = a$, or $y + 1 = a$.

Hence, since in the above mentioned equation

$$\frac{(y+1)b-r}{a}=x', \text{ or}$$

$$(y+1)b-ax'=r,$$

$y+1$ is less then a, x' must necessarily be less than b; and, consequently,

$$x=b-1-\frac{(y+1)b-r}{a}=b-1-x'.$$

Where x' being less than b, it follows, that x, or its equal $b-1-x'$, must be either $=0$, or to some integer number, when a, b, and c are prime to each other.

Whence the equation $ax+by=c$ is always possible, under the proposed conditions.

Cor. The above proposition, as well as the former, is of great use in enabling us to judge of the possibility or impossibility of indeterminate equations of this kind; and, consequently, also in proposing them so that they may be within possible limits (*l*).

OF THE DIOPHANTINE ANALYSIS.

(G) P. 244. The Diophantine analysis has been so fully treated of in the former part of the present work, that nothing farther need be here said on the subject; except to observe, that the solutions of

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(*l*) This theorem arises out of a remark, which appears to have been first made by Waring, in his Meditationes Algebraicæ, p. 340, third edition; but in what way he convinced himself of its truth does not appear, as a direct investigation of it cannot be easily obtained.

The above demonstration is similar to that given by Mr. Barlow in his Elementary Investigation of the Theory of Numbers, p. 77.

many questions that may occur in this branch of the science must frequently be left, notwithstanding the various rules that have been given for this purpose, to the skill and sagacity of the learner, who, in pursuing these inquiries, will soon perceive, that nothing less than the most refined algebra, applied with the utmost skill and judgement, can surmount the various difficulties which attend them; and, in this way, no one has ever extended the limits of the analytic art farther than Diophantus, or discovered greater knowledge and penetration in the application of it.

When we consider his work with attention, we are at a loss which to admire most, his wonderful sagacity, and the peculiar artifices he employs, in forming such positions as the nature of the problems required, or the more than ordinary subtilty of his reasoning upon them.

Every particular question puts us upon a new way of thinking, and furnishes a fresh vein of analytical treasure, which cannot but prove highly instructive to the mind, in conducting it through almost all difficulties of this kind, whenever they occur (*m*).

(*m*) That Diophantus was not the inventor of algebra, as has been generally imagined, is obvious; for his method of applying it is such as could only have been used in a very advanced state of the science. He no where speaks of the fundamental rules and principles, as an inventor certainly would have done, but treats of it as an art already sufficiently known; and seems to intend, not so much to teach it, as to cultivate and improve it, by solving such questions as, before his time, had been thought too difficult to be surmounted.

It is highly probable, therefore, that algebra was known

OF CONTINUED FRACTIONS.

(H) P. 282. In addition to what has been said on the subject of the present article, it may here be farther observed, that we can readily find the series of fractions converging towards any given quantity, without being under the necessity of first reducing it to the continued form.

For, since it has been shown, in the practical article (H), above referred to, that the denominators α, β, γ, δ, &c. of the continued fraction

$$\frac{a}{b}=\alpha+\frac{1}{\beta+\frac{1}{\gamma+\frac{1}{\delta+\frac{1}{\epsilon\,\&c.}}}},$$

are the quotients obtained by finding the common measure of the numerator and denominator of the given fraction $\frac{a}{b}$, it is obvious that we can readily

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among the Greeks long before the time of Diophantus; but that the works of preceding writers have been destroyed by the ravages of time, or the depredations of war and barbarism.

His *Quest. Arithmet.*, out of which these problems are mostly collected, consisted originally of thirteen books; but the first six only are now extant: the best edition of which is commonly thought to be that published at Paris, by Bachet, in the year 1670.

In this performance, the subject is so skilfully handled, that the moderns, notwithstanding their other improvements, have been able to do little more than explain and illustrate his method. Those who have succeeded best, in this respect, are Kersey, De Billy, Ozanam, Prestet, Saunderson, Fermat, and Euler; the last of whom, in particular, has amplified and generalised this curious branch of the science, in as full and satisfactory a manner as the nature of the subject seems to admit.

ascertain the series of converging fractions, answering to these quotients, by means of the law of their generation, pointed out in Nos. 3 and 4 of that article, without having recourse to any intermediate operation.

So that in fact, the consideration of any quantity, whether rational or irrational, under the form of a continued fraction, is no otherwise necessary than as it leads us to the properties and law of formation, of the several converging fractions which are derived from them; it being in this last form only, that these expressions are at all applicable to any useful purpose.

1. Thus, in order to repass from the continued fraction to the common fraction from which it was derived, we shall have the following results, or converging fractions; as has been before shown.

$$\alpha = \frac{\alpha}{1},\ \alpha + \frac{1}{\beta} = \frac{\alpha\beta + 1}{\beta},\ \alpha + \cfrac{1}{\beta + \cfrac{1}{\gamma}} = \frac{\alpha\beta\gamma + \gamma + \alpha}{\beta\gamma + 1},$$

$$\alpha + \cfrac{1}{\beta + \cfrac{1}{\gamma + \cfrac{1}{\delta}}} = \frac{\alpha\beta\gamma\delta + \gamma\delta + \alpha\delta + \alpha\beta + 1}{\beta\gamma\delta + \delta + \beta}.$$

Or, taking $\frac{1}{0}$ for the first term, and placing them according to the form in which they stand in the first part of the work, we shall have

$$\begin{array}{cccccc} \alpha & \beta & \gamma & \delta & \epsilon & \\ \frac{1}{0}, & \frac{\alpha}{1}, & \frac{\alpha\beta + 1}{\beta}, & \frac{\alpha\beta\gamma + \gamma + \alpha}{\beta\gamma + 1}, & \frac{\alpha\beta\gamma\delta + \gamma\delta + \alpha\delta + \alpha\beta + 1}{\beta\gamma\delta + \delta + \beta}, & \&c. \end{array}$$

Where it is evident, that each numerator, of the fractions thus obtained, is equal to the product of the preceding numerator and the quotient standing

over it, plus the numerator of the antepenultimate fraction; and the denominators are formed according to the same law.

So that, if $\frac{p^0}{q^0}$, $\frac{p}{q}$, and $\frac{p'}{q'}$, be any three consecutive fractions in the above series, and μ be the quotient standing over $\frac{p}{q}$, we shall have,

$$\frac{p'}{q'}=\frac{p\mu+p^0}{q\mu+q^0}$$

And if, agreeably to what has been said in the practical article, Vol. I, there be substituted, instead of μ, the complete quotient $\mu+\frac{y}{z}$, we shall have

$$\frac{a}{b}=\frac{p(\mu+\frac{y}{z})+p^0}{q(\mu+\frac{y}{z})+q^0}$$

Which is, manifestly, the exact value of the original fraction.

Also, if the quantity $\frac{a}{b}$, that is to be converted into a continued fraction, be a rational fraction, it is obvious that this will be the last of the converging fractions above given; since the series must necessarily terminate at this point.

But if the quantity x that is to be developed into a similar series,

$$x=\alpha+\frac{1}{\beta+\frac{1}{\gamma+\frac{1}{\delta+\frac{1}{\varepsilon\,\&c.}}}}$$

be irrational, or transcendental, the continued fraction may be produced without end; and, consequently, the number of converging fractions will, in this case, be indefinite.

2. It may here, also, be farther shown, that the several terms of any series of converging fractions are alternately less and greater than the total value x, of the continued fraction from which they are derived.

For, if we consider only the first term of the series, it is evident, that

$$x \asymp \alpha, \text{ or } \alpha \asymp x.$$

And, if the two first terms of the series be taken, it follows, since the denominator β is less than when the sum of the following terms is joined to it, that

$$\alpha + \frac{1}{\beta} \asymp x;$$

it being obvious, that the less the denominator of any proper fraction is, the greater will be the fraction; and the contrary.

Also, because the denominator γ, in the sum of the three first terms, is less than when the rest of the terms are joined to it, the fraction $\frac{1}{\gamma}$ will be too great; and, consequently, $\beta + \frac{1}{\gamma}$ too little; or

$$\alpha + \frac{1}{\beta + \frac{1}{\gamma}} \asymp x.$$

And so on, alternately, for all the rest of these fractions.

3. If the terms of any two consecutive converging fractions, $\frac{p^\circ}{q^\circ}$ and $\frac{p}{q}$, be multiplied crosswise, we shall have

$$pq^\circ - qp^\circ = \pm 1,$$

taking the upper sign $+1$, if $\frac{p}{q}=x$, or when it is one of the odd terms of the series, reckoning from $\frac{1}{0}$, and -1, if the same fraction be $=x$, or when it is one of the even terms of the series.

For let there be taken the following series of quotients, and their corresponding converging fractions,

$$\begin{array}{cccccccccc} \alpha & \beta & \gamma & & \mu^\circ & \mu & \mu' & \mu'' & \&c. \\ \frac{1}{0}, & \frac{\alpha}{1}, & \frac{\alpha\beta+1}{\beta} & \ldots\ldots & \frac{p^\circ}{q^\circ}, & \frac{p}{q}, & \frac{p'}{q'}, & \frac{p''}{q''}, & \&c. \end{array}$$

Then we shall have, by what has been before shown,

$$p''=p'\mu'+p, \text{ and } q''=q'\mu'+q, \text{ or}$$
$$\mu'=\frac{p''-p}{p'}=\frac{q''-q}{q'}.$$

Or, by clearing the two last equalities of fractions, and transposing the terms,

$$q'p''-p'q''=pq'-qp'.$$

And because, for a reason similar to that above mentioned,

$$p'=p\mu+p^\circ, \text{ and } q'=q\mu+q^\circ; \text{ or}$$
$$\mu=\frac{p'-p^\circ}{p}=\frac{q'-q^\circ}{q},$$

we shall readily obtain, in the same manner as before,

$$qp'-pq'=p^\circ q-q^\circ p.$$

Hence, comparing this equation with the former, we shall have

$$q'p''-p'q''=pq'-qp'=pq^\circ-qp^\circ;$$

which equalities, it is manifest, will hold for any three successive terms of the series.

Whence, the differences between the products of the numerators and denominators of any two consecutive fractions, in that series, abstracting from their signs, are equal to each other.

But the difference between those of the first two fractions, $(\alpha\beta + 1) - \alpha\beta$, is manifestly 1; and, therefore, since the differences are all equal, they are each $= 1$; and, consequently,

$$pq^\circ - qp^\circ = \pm 1;$$

taking the upper sign $+$, or the under sign $-$, according to the rule mentioned in the proposition.

Cor. It may here be added, that it is this property of converging fractions, that renders them so useful in the resolution of indeterminate equations of the first degree; since every equation of this kind has its solution depending upon that of the equation

$$ax + by = \pm 1;$$

as has been already shown in the corresponding article of the first part of the present work.

It may also be readily concluded, from the above proposition, that every converging fraction $\frac{p^\circ}{q^\circ}$, $\frac{p}{q}$, &c. in the before mentioned series, is in its lowest, or most simple terms.

For, if p° and q°, or p and q, had any common divisor, except unity, it would follow, from the property of the equation

$$pq^\circ - qp^\circ = \pm 1;$$

that 1 must also be divisible by that divisor, which is impossible.

4. Any converging fraction, in the series before given, approaches nearer to the true value of the original fraction than the one that immediately precedes it; and the error arising from taking any one of these fractions for the total value of the continued fraction, is always less than unity divided by the square of the denominator of that fraction.

For let $\frac{p^\circ}{q^\circ}$, $\frac{p}{q}$, and $\frac{p'}{q'}$, be any three consecutive fractions, in the abovementioned series, and μ the quotient corresponding with $\frac{p}{q}$; then, as before, we shall have

$$\frac{p'}{q'} = \frac{p\mu + p^\circ}{q\mu + q^\circ}.$$

And if $\mu + \frac{y}{z}$, or, for the sake of simplicity, y', be put in the place of μ, the whole continued fraction will be represented by

$$x = \frac{py' + p^\circ}{qy' + q^\circ}.$$

And, consequently, by subtracting $\frac{p}{q}$ and $\frac{p^\circ}{q^\circ}$ from each side of this last expression, we shall have

$$x - \frac{p}{q} = \frac{qp^\circ - pq^\circ}{q(qy' + q^\circ)} = \mp \frac{1}{q(qy' + q^\circ)},$$

$$x - \frac{p^\circ}{q^\circ} = \frac{(pq^\circ - qp^\circ)y'}{q^\circ(qy' + q^\circ)} = \pm \frac{y'}{q^\circ(qy' + q^\circ)};$$

where it appears, that $x - \frac{p}{q}$ and $x - \frac{p^\circ}{q^\circ}$ have always different signs; and, consequently, as before shown,

the true value of x will fall between any two consecutive fractions in the abovementioned series.

Also, since from the two last of these equations we have

$$p-qx=\pm\frac{1}{qy'+q^\circ},\text{ and }-p^\circ+q^\circ x=\pm\frac{y'}{qy'+q^\circ},$$

it follows, from the denominators of the fractional parts of these expressions being equal, that

$$\frac{p-qx}{-p^\circ+q^\circ x}=\frac{1}{y'}.$$

But, from the nature of these fractions, y', or its equal $\mu+\frac{y}{z}$, is a number greater than unity; whence

$$\frac{p-qx}{q^\circ x-p^\circ}=1,\text{ or } p-qx=q^\circ x-p^\circ.$$

And because, for the same reason, $q^\circ=q$, much more must

$$\frac{p}{q}-x=x-\frac{p^\circ}{q^\circ}.$$

Hence, since the difference between any converging fraction and the original fraction, is less than the difference between that which immediately precedes it and the original one, it is evident that any fraction $\frac{p}{q}$, in the series, approaches nearer to the true value x, of the continued fraction, than any of the preceding ones.

In like manner, if μ be a number next less than y', or such that

$$y'=\mu,\text{ and } y'=\mu+1,$$

it follows, as a necessary consequence, that

$$qy'+q^\circ=q\mu+q^\circ,\text{ and } qy'+q^\circ=q(\mu+1)+q^\circ.$$

And, consequently, from what has been before shown, we shall have, in this case,

$$x-\frac{p}{q}=\mp\frac{1}{q(q\mu+q^{\circ})},\text{ or }=\frac{1}{qq'}.$$

And

$$x-\frac{p}{q}=\mp\frac{1}{q[q(\mu+1)+q^{\circ}]},\text{ or }=\frac{1}{q(q'+q)}.$$

From which it is evident, that the difference between any fraction in the series, and the original fraction, is always less than unity divided by the product of the denominators of that fraction and the one immediately following it; but greater than unity divided by the sum of the same product and the square of the denominator of the fraction in question.

Or, since the quotients α, β, γ, &c. are all whole positive numbers, we shall have, as before observed,

$$q^{\circ}=q,\text{ and }q=q';\text{ or }qq'=q^2,\text{ and }\frac{1}{qq'}=\frac{1}{q^2}.$$

And, consequently, by comparing this value with that above mentioned, it follows, that much more must

$$x-\frac{p}{q}=\pm q^{2}.$$

Whence, the error arising from taking any of these fractions for the total value of the continued fraction is always less than unity divided by the square of the denominator of that fraction.

5. Any converging fraction $\frac{p}{q}$, or $\frac{p'}{q'}$, expresses the total value of the continued, or original fraction, x, more accurately than any other fraction $\frac{m}{n}$, that has a less denominator, or which can be conceived in more simple terms.

For since, as before shown, x falls between $\frac{p}{q}$ and $\frac{p'}{q'}$, if $\frac{m}{n}$ can also be inserted between the same two fractions, we shall have, abstracting from the sign,

$$\frac{p}{q}-\frac{m}{n}\gtrless\frac{p}{q}-x, \text{ or } \gtrless\frac{1}{q^2}.$$

Or, bringing the first member of this expression to a common denominator,

$$\frac{pn-qm}{qn}\gtrless\frac{1}{q^2}.$$

But p, q, m, n, being whole numbers, the numerator, $pn-qm$, cannot be less, abstracting from its sign, than unity; and therefore, since, by hypothesis, $p\gtrless q$, we shall also have $qn\gtrless q^2$; and, consequently,

$$\frac{pn-qm}{qn}\gtrless\frac{1}{q^2}.$$

Which being an opposite result to the former, it follows that $\frac{m}{n}$ cannot be inserted between $\frac{p}{q}$ and x.

Again, let, if possible, $\frac{m}{n}$ be nearer to x than $\frac{p'}{q'}$; then we shall have, as before,

$$\frac{p'}{q'}-\frac{m}{n}\gtrless\frac{p'}{q'}-x, \text{ or } \gtrless\frac{1}{q'^2}.$$

Or, bringing, as in the former instance, the first member of the expression to a common denominator,

$$\frac{p'n-q'm}{q'm}\gtrless\frac{1}{q'^2}.$$

But this cannot take place, because, for a reason similar to that above given, the numerator cannot be $\gtrless 1$; and n is $\gtrless q'$, or $q'n\gtrless q'^2$.

Hence, since the fraction $\frac{m}{n}$ cannot be inserted between $\frac{p}{q}$ and $\frac{p'}{q'}$, or between $\frac{p'}{q'}$ and $\frac{p''}{q''}$, &c. it is obvious that $\frac{p}{q}$, or $\frac{p'}{q'}$, &c. expresses the value of x more exactly than $\frac{m}{n}$.

6. With respect to the principal converging fractions $\frac{A}{A'}$, $\frac{B}{B'}$ &c. . . . $\frac{P^0}{Q^0}$, $\frac{P}{Q}$, $\frac{P'}{Q'}$ &c. using the notation employed in the former part of the work, it is plain, since they are alternately less and greater than the total value x of the continued fraction, that we can separate them into two classes,

$$\frac{A}{A'}, \quad \frac{C}{C'}, \quad \frac{E}{E'}, \quad \frac{G}{G'} \text{ \&c.}$$

$$\frac{B}{B'}, \quad \frac{D}{D'}, \quad \frac{F}{F'}, \quad \frac{H}{H'} \text{ \&c.}$$

the former of which is composed of fractions that are all less than x, but approaching nearer to it at every term; and the latter of fractions all greater than x, but each of them coming nearer to it, the farther they are continued.

Hence, agreeably to what has been shown in the preceding articles, we shall have, for the first class,

$$\frac{C}{C'} - \frac{A}{A'} = \frac{CA' - AC'}{A'C'} = \frac{A'(B\gamma + A) - A(B'\gamma + A')}{A'C'} = \frac{\gamma}{A'C'}$$

$$\frac{E}{E'} - \frac{C}{C'} = \frac{EC' - E'C}{C'E'} = \frac{C'(D\varepsilon + C) - C(D'\varepsilon + C')}{C'E'} = \frac{\varepsilon}{C'E'}$$

&c. &c.

And for the second,

$$\frac{B}{B'}-\frac{D}{D'}=\frac{BD'-B'D}{B'D'}=\frac{B(C'\delta+B')-B'(C\delta+B)}{B'D'}=\frac{\delta}{B'D'}$$

$$\frac{D}{D'}-\frac{F}{F'}=\frac{DF'-FD'}{F'D'}=\frac{D(E\zeta+D')-D'(E\zeta+D)}{F'D'}=\frac{\zeta}{F'D'}$$

&c. &c.

Where it is evident, when the numbers γ, δ, ε &c. denoting the several quotients, are all equal to unity, that it will be impossible to insert between any two consecutive fractions, in either of these series, any other fraction whose denominator shall fall between those of the fractions in question.

But this will not be the case when the numbers γ, δ, ε &c. are greater than unity; for suppose, for example, that $\gamma=4$; then there will arise

$$C=4B+A,\quad C'=4B'+A.$$

And, consequently, in this case, we can insert between the fractions $\frac{A}{A'}$, $\frac{C}{C'}$, the three intermediate fractions,

$$\frac{B+A}{B'+A'},\quad \frac{2B+A}{2B'+A'},\quad \frac{3B+A}{3B'+A'}$$

But since the denominators of these fractions increase, by equal differences, from A' to C', and the numerators from A to C, the fractions themselves will also increase from $\frac{A}{A'}$ to $\frac{C}{C'}$, in such a manner, that it will be impossible to insert between any two consecutive fractions in the series

$$\frac{A}{A'},\quad \frac{B+A}{B'+A'},\quad \frac{2B+A}{2B'+A'},\quad \frac{3B+A}{3B'+A'},\quad \frac{4B+A}{4B'+A'}=\frac{C}{C'},$$

any other fraction, whose denominator shall fall between the denominators of the same two fractions.

For, if we take the differences between each of

the succeeding and preceding fractions, we shall have, on account of $BA' - AB' = 1$,

$$\frac{A+B}{B+A'} - \frac{A}{A'} = \frac{1}{A'(B'+A')}$$

$$\frac{2B+A}{2B'+A'} - \frac{B+A}{B'+A'} = \frac{1}{(B'+A')(2B'+A')}$$

$$\frac{3B+A}{3B'+A'} - \frac{2B+A}{2B'+A'} = \frac{1}{(2B'+A')(3B'+A')}$$

$$\frac{C}{C'} - \frac{3B+A}{3B'+A'} = \frac{1}{(3B'+A')C'}$$

From which it is evident that the fractions $\frac{A}{A'}$, $\frac{B+A}{B'+A'}$ &c. continually increase, because their differences are all positive.

Also, because these differences are equal to unity divided by the product of the two denominators, it is impossible to interpolate any fraction whatever between two consecutive fractions in the above series, whose denominator shall fall between the denominators of those fractions.

Furthermore, since the fractions in question are all less than the true value of x, and the fraction $\frac{B}{B'}$ is greater, it is evident that each of these fractions wlll approach nearer to x than $\frac{B}{B'}$; for, as before,

$$\frac{A}{A'} - \frac{B}{B'} = \frac{-1}{A'B'}$$

$$\frac{B+A}{B'+A'} - \frac{B}{B'} = \frac{-1}{B'(B'+A')}$$

$$\frac{2B+A}{2B'+A'} - \frac{B}{B'} = \frac{-1}{B'(2B'+A')}$$

$$\frac{3B+A}{3B'+A'} - \frac{B}{B'} = \frac{1}{B'(3B'+A')}$$

$$\frac{C}{C'} - \frac{B}{B'} = \frac{-1}{B'C'}$$

Hence, since these differences are also equal to unity divided by the product of the denominators, it may be proved, in the same manner as before, that no fraction can be interpolated between either of the fractions $\frac{A}{A'}$, $\frac{B+A}{B'+A'}$, $\frac{2B+A}{2B'+A'}$, &c. and $\frac{B}{B'}$, whose denominator shall fall between the denominators of these two fractions; and so on for any other terms

$$\frac{C}{C'}, \frac{E}{E'}, \&c.$$

In like manner, when the quotients δ, ξ, &c. are each greater than unity, we can insert between the terms $\frac{B}{B'}$, $\frac{D}{D'}$, and $\frac{D}{D'}$, $\frac{F}{F'}$, of the second series, different intermediate fractions, that shall be all greater than x, but approaching it more nearly the farther they are continued.

Also, since β is a number greater than unity, we can place before the fraction $\frac{B}{B'}$ the several fractions $\frac{A+1}{1}$, $\frac{2A+1}{2}$, $\frac{3A+1}{3}$ &c. as far as to $\frac{\delta A+1}{\delta}=\frac{B}{B'}$, which shall all have the same properties as the intermediate fractions.

Hence, by proceeding in this manner, we shall have the two following series of fractions converging towards the quantity x, which represents the total value of the continued fraction.

Increasing Fractions less than x.

$$\begin{array}{llll}
\frac{A}{A'}, & \frac{B+A}{B'+A'}, & \frac{2B+A}{2B'+A'} \cdots\cdots & \frac{(\gamma-1)B+A}{(\gamma-1)B'+A'} \\
\frac{C}{C'}, & \frac{D+C}{D'+C'}, & \frac{2D+C}{2D'+C'} \cdots\cdots & \frac{(\varepsilon-1)D+C}{(\varepsilon-1)D'+C'} \\
\frac{E}{E'}, & \frac{F+E}{F'+E'}, & \frac{2F+E}{2F'+E'} \cdots\cdots & \frac{(\eta-1)F+E}{(\eta-1)F'+E'} \\
\&c. & & & \&c.
\end{array}$$

Decreasing Fractions greater than x.

$$\frac{A+1}{1},\quad \frac{2A+1}{2},\quad \frac{3A+1}{3}\cdots\cdots\frac{(\beta-1)A+1}{\beta-1}$$

$$\frac{B}{B'},\quad \frac{C+B}{C'+B'},\quad \frac{2C+B}{2C'+B'}\cdots\cdots\frac{(\delta-1)C+B}{(\delta-1)C'+B'}$$

$$\frac{D}{D'},\quad \frac{E+D}{E'+D'},\quad \frac{2E+D}{2E'+D'}\cdots\cdots\frac{(\zeta-1)E+D}{(\zeta-1)E'+D'}$$

&c. &c.

Each of which approaches nearer to the true value of x than any other fraction having a less denominator.

If the quantity x be irrational, the two preceding series may be evidently continued without end, because the principal fractions $\frac{A}{A'}$, $\frac{B}{B'}$ &c. proceed of themselves to infinity.

But if x be rational, or equal to any fraction whatever, $\frac{a}{b}$, the series of principal fractions, in the latter of these cases, will terminate, the last of them being $\frac{a}{b}$; but the other series will go on to infinity.

7. The operation made use of in Art. (H) No. 17, of the former part of the work, for converting the square root of 19, and other nonquadrate numbers, into converging fractions, may be rendered much more simple, by observing the following law of the formation of the successive quotients, *viz.* let

$$\frac{\sqrt{19}+m}{n}=u+\&c.\ \text{and}\ \frac{\sqrt{19}+m'}{n'}=u'+\&c.$$

represent any two consecutive fractions in the example there given, u, u' being their respective quotients.

Then, from the nature of the operation, we shall have

$$mu - m = m' \text{ and } \frac{19-(nu-m)^2}{n} = n'.$$

So that each value of m', n', and u', may be readily deduced from those of m, n, and u, in the preceding fraction. Hence the operation above referred to, will stand, by means of this law, as follows:

$\frac{\sqrt{19}+0}{1} = 4 + \&c.$	$\frac{19-4^2}{1} = 3$
$\frac{\sqrt{19}+4}{3} = 2 + \&c.$	$\frac{19-2^2}{3} = 5$
$\frac{\sqrt{19}+2}{5} = 1 + \&c.$	$\frac{19-3^2}{5} = 2$
$\frac{\sqrt{19}+3}{2} = 3 + \&c.$	$\frac{19-3^2}{2} = 5$
$\frac{\sqrt{19}+3}{5} = 1 + \&c.$	$\frac{19-2^2}{5} = 3$
$\frac{\sqrt{19}+2}{3} = 2 + \&c.$	$\frac{19-4^2}{3} = 1$
$\frac{\sqrt{19}+4}{1} = 8 + \&c.$	$\frac{19-4^2}{1} = 3$

Where it will be found, by continuing the process, that the period of quotients 2, 1, 3, 1, 2, 8, will recur again in the same order, ad infinitum.

Hence, placing these quotients as in the former part of the work, and following a similar mode of operation, we shall have the following series of converging fractions:

$$\begin{array}{ccccccc} 4, & 2, & 1, & 3, & 1, & 2, & 8 \\ \frac{1}{0}, & \frac{4}{1}, & \frac{9}{2}, & \frac{13}{3}, & \frac{48}{11}, & \frac{61}{14}, & \frac{160}{39} \end{array} \&c.$$

each of which expresses $\sqrt{19}$ more nearly than any preceding one; and it is evident that they may b

continued at pleasure, to any degree of accuracy required.

Hence, also, the square root of any nonquadrate number N, may be extracted in the same manner, by supposing a to be the greatest integer contained in $\sqrt{N}$, and u, u', u'', &c. the greatest integers contained in the several fractions with which they correspond; *viz.*

$$\frac{\sqrt{N}+0}{1}=a+\&c \qquad \frac{N-(1\times a-0)^2}{1}=n$$

$$\frac{\sqrt{N}+m}{n}=u+\&c. \qquad \frac{N-(nu-m)^2}{n}=n'$$

$$\frac{\sqrt{N}+m'}{n'}=u'+\&c. \qquad \frac{N-(n'u'-m')^2}{n'}=n''$$

$$\frac{\sqrt{N}+m''}{n''}=u''+\&c. \qquad \frac{N-(n'u''-m'')^2}{n''}=n'''$$

&c. &c.

Where, by continuing the extraction, as in the former case, we shall always arrive at a fraction equal to $\frac{\sqrt{N}+m}{n}$; after which, the quotient will constantly recur again in the same order.

It is also evident, from the practical example before given, and may be demonstrated generally, that the last quotient of every complete period of quotients, in this mode of extracting the square root of any number, is always equal to $2a$, supposing a to be the greatest integer contained in $\sqrt{N}$.

We might here also have proceeded to show in what way the solutions of the indeterminate equations

$$x^2-Ay^2=\pm 1, \text{ and } x^2-Ay^2=\pm B,$$

of the second degree, are derived from the principles above demonstrated; but the details, necessary

for this purpose, are too long and complicated to be given in a confined work like the present (*n*).

OF THE DECOMPOSITION OF RATIONAL FRACTIONS INTO SIMPLE FRACTIONS.

(1) P. 312. The method of decomposing rational fractions into simple fractions, has been so fully explained in the corresponding article of the former part of the work, that nothing more need be said on the subject, except to refer the student, who may be desirous of seeing this important branch of analysis treated at greater length, to that excellent work, the *Introductio in Analysin Infinitorum*, of Euler, where he will find both the theory and practice of the whole of this doctrine clearly investigated and exemplified.

He may also consult, for the same purpose, Simpson's Treatise of Fluxions, sect. 4 and 5, vol. ii, and his Essays, p. 118; as well as the *Traité de calcul intégral* of Bougainville, and the more recent works on the same subject by Cousin and Lecroix; in each of which both the algebraical and fluxional method of treating this branch of the science, together with their most useful applications, are detailed with great perspicuity and elegance.

(*n*) The reader, who may wish to see a full account of this subject, is referred to the Theory of Numbers, by Legendre, so often mentioned; the Memoirs of the Academy of Berlin, An. 1767 and 1768; and to the second volume of Euler's Algebra, English edition; where he will find the whole doctrine of continued fractions, and every part of the indeterminate analysis, amply developed.

OF RECURRING SERIES.

(K) P. 320. THE subject of Recurring Series, treated of in article (K) in the first part of the present work, is a branch of the general doctrine of series, hereafter mentioned, for which we are indebted to de Moivre; who appears to have been led to this useful discovery by his inquiries into the calculations of probabilities, or the doctrine of Chances; and which he afterwards made a part of his treatise entitled *Miscellanea analytica de Seriebus et quadraturis* (London, 1730); a work which abounds with the most ingenious and profound researches in several of the more difficult branches of analysis.

Expressions of this kind are derived, as before observed, from the expansion of some rational fraction of the form

$$\frac{a + bx + cx^2 + dx^3 + \ldots\ldots + px^{m-1}}{1 - a'x - b'x^2 - c'x^3 - \ldots\ldots - q'x^m}$$

into a series, in which the coefficient of any one of the terms, beginning with the mth, is derived from those of the m preceding terms, according to a certain law, which is determined by the denominator of the given fraction (o).

(o) In every case of this kind, the highest power of the unknown quantity in the numerator of the generating fraction, must be less than that in the denominator, or otherwise the development of it will not give a recurring series. Hence, when the fraction is not in this state, it must be reduced to it, by dividing the numerator by the denominator, till the remainder fall under the conditions required.

Thus, the following rational fractions,

$$\frac{a}{1-a'x}, \quad \frac{a+bx}{1-a'x-b'x^2}, \quad \frac{a+bx+cx^2}{1-a'x-b'x^2-c'x^3}, \text{ \&c.}$$

are expressed by a recurring series of the form

$$A + A_1x + A_2x^2 + A_3x^3 + A_4x^4 \text{ \&c.};$$

where, for the first of these fractions, we have $A = a$, the scale of relation being here composed of a single term a'; and each of the following terms is equal to the preceding term multiplied by a'.

For the second fraction, the scale of relation is composed of the two terms a', b'; and in this case the coefficient of each term is equal to the sum of the coefficients of the two preceding terms multiplied by a' and b' respectively; and so on for any other fraction.

So that the coefficients of the first and other succeeding powers of the unknown quantity, in the denominator, with their signs changed, form the scale of relation of the developed series; and, vice versa, the first, second, &c. terms of the scale of relation, are equal to the second, third, &c. terms of the denominator, divided by the first, and taken with a contrary sign.

Thus, for example, if the fraction be $\frac{1+x^3}{1+3x+x^2}$, we shall have, by dividing x^3+1 by x^2+3x+1,

$$\frac{x^3+1}{x^2+3x+1} = x - 3 + \frac{4+8x}{1+3x+x^2}$$

where the remainder, $\frac{4+8x}{1+3x+x^2}$, when developed, will give the recurring series

$$4 - 4x + 16x^2 - 44x^3 + 116x^4 - 294x^5 + \text{ \&c.}$$

Hence, the scale of relation of any recurring series being given, the denominator of the generating fraction becomes known; and consequently it only remains to determine the numerator, in order to obtain the fraction itself, or the whole sum of the series.

For this purpose, therefore, let there be proposed the general recurring series

$$A + Bx + Cx^2 + Dx^3 + Ex^4 \text{ \&c.}$$

and suppose the scale of relation, as before mentioned, to be

$$a' - b', \quad + c'; \quad - d';$$

then, from what has been above observed, the denominator of the generating fraction will be

$$1 - a'x + b'x^2 - c'x^3 + d'x^4.$$

Hence, assuming $a + bx + cx^2 + dx^3$ for the numerator, we shall have the fraction

$$\frac{a + bx + cx^2 + dx^3}{1 - a'x + b'x^2 - c'x^3 + d'x^4}$$

And, consequently, by comparing this with the series $A + Bx + Cx^2 + Dx^3 + Ex^4$ &c. there will arise the identity

$$a + bx + cx^2 + dx^3 = (A + Bx + Cx^2 + Dx^3 + \text{\&c.}) \times$$
$$(1 - a'x + b'x^2 - c'x^3 + d'x^4)$$

Which, by actually multiplying the two last factors, and comparing the coefficients of the like powers of x, gives

$$a = A$$
$$b = B - a'A$$
$$c = C - a'B + b'A$$
$$d = D - a'C + b'B - c'A$$
$$\text{\&c.} \qquad \text{\&c.}$$

Consequently the generating fraction, or entire sum of the series, continued to infinity, will be

$$\frac{A + (B - a'A)x + (C - a'B + b'A)x^2 + (D - a'C + b'B - c'A)x^3}{1 - a'x + b'x^2 - c'x^3 + d'x^4}$$

From which last expression, we may now readily find the sum of any number of terms of a recurring series of this kind.

For, suppose, for the sake of greater simplicity, the scale of relation of the series, above proposed, to be, in this case,

$$a', \quad -b', \quad +c'.$$

Then, if it were required to find the sum of its n first terms, we shall have,

$$S = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c. \textit{ ad inf.}$$

And for the sum of all the terms, following Px^n, to infinity,

$$S' = Qx^{n+1} + Rx^{n+2} + Sx^{n+3} + Tx^{n+4} + \&c.$$

Whence, dividing this last series by x^{n+1}, or making it a common factor to the rest of the terms, there will arise a recurring series, perfectly conformable with the first; the sum of which, agreeably to the present scale of relation, will be

$$S' = \frac{Qx^{n+1} + (R - a'Q)x^{n+2} + (S - a'R + b'Q)x^{n+3}}{1 - a'x + b'x^2 - c'x^3}.$$

And, therefore, the difference, $S - S'$, between this expression and the former, or the sum of the n first terms of the series, will be,

$$\frac{A + (B - a'A)x + (C - a'B + b'A)x^2}{1 - a'x + b'x^2 - c'x^3}$$

$$- \frac{Qx^{n+1} + (R - a'Q)x^{n+2} + (S - a'R + b'Q)x^{n+3}}{1 - a'x + b'x^2 - c'x^3}.$$

Which mode of proceeding may be applied to

any other case, where the scale of relation consists of a greater number of terms.

To this we may farther add the following method of finding the sum of any series of this kind, as given by Simpson in his Essays, page 96.

Let $A+B+C+D+\ldots+K+L+M+N$, be any recurring series, of which each term depends upon the three that precede it; the scale of relation being

$$p,\quad q,\quad r,\quad s.$$

Then, by the nature of the proposed series, we shall have the following equations:

$$\begin{aligned}
pA+qB+rC+sD&=0\\
pB+qC+rD+sE&=0\\
pC+qD+rE+sF&=0\\
pD+qE+rF+sG&=0\\
&\ldots\ldots\\
pK+qL+rM+sN&=0
\end{aligned}$$

And, consequently, by taking the sum of these last equations, there will arise

$$\left.\begin{aligned}p(A+B+C+\ldots+K)+q(B+C+D+\ldots+L)\\+r(C+D+\ldots+M)+s(D+\ldots\ldots+N)\end{aligned}\right\}=0$$

Whence, if s be made to denote the sum of all the terms of the proposed series, we shall have

$$\left.\begin{aligned}p(S-L-M-N)+q(S-A-M-N)\\+r(S-A-B-N)+s(S-A-B-C)\end{aligned}\right\}=0$$

And, consequently, by addition and division,

$$S=\frac{p(L+M+N)+q(A+M+N)+r(A+B+N)+s(A+B+C)}{p+q+r+s}$$

Where it is evident, that the sum required depends solely upon the three first terms of the series and the three last.

And a similar mode of investigation will apply when the scale of relation consists of a great er. number of terms.

The same method may also be used, in determining the fraction from which the proposed series is derived.

For supposing the series to be infinite, the number of equations

$$pA + qB + rC + sD = 0$$
$$pB + qC + rD + sE = 0$$
$$pC + qD + rE + sF = 0$$
$$pD + qE + rF + sG = 0$$
$$\&c.$$

upon this hypothesis, will become indefinite; and, by adding them together, we shall have,

$$\left.\begin{array}{l} p(A + B + C + D + \&c.) + q(B + C + D + \&c.) \\ + r(C + D + \ldots + \&c.) + s(D + \ldots + \&c.) \end{array}\right\} = 0$$

Which, by putting $S = A + B + C + D + \&c.$ continued to infinity, becomes

$$pS + q(S - A) + r(S - A - B) + s(S - A - B - C) = 0.$$

Whence, by multiplying and collecting the several terms of this last expression, we have

$$S = \frac{A(q + r + s) + B(r + s) + C}{p + q + r + s}.$$

Thus, for example, let there be taken the recurring series,

$$1 + 2x + 8x^2 + 28x^3 + 100x^4 + \&c.$$

of which each term is formed from the two preceding ones, multiplied respectively by $2x^2$ and $3x$; then

$$A = 1,\ B = 2x,\ C = 8x^2,\ D = 28x^3,\ \&c.$$

And, consequently,

$$\left.\begin{array}{l} C = 2x^2A + 3xB \\ D = 2x^2B + 3xC \end{array}\right\} \text{ Or } \left\{\begin{array}{l} -2x^2A - 3xB + C = 0 \\ -2x^2B - 3xC + D = 0 \end{array}\right.$$

&c. &c.

From which it appears that the scale of relation is

$$p = -2x^2,\ q = -3x,\ r = 1,\ s = 0.$$

And, therefore,

$$s = \frac{1 - x}{-2x^2 - 3x + 1} = \frac{x - 1}{2x^2 + 3x - 1},$$

Which is the rational fraction sought; as will appear from its development giving the series first proposed (*p*).

With respect to the method of finding the general term of a series of this kind, it will be proper to take, in the first place, the simple rational fraction,

$$\frac{A}{1 - rx}.$$

Of which the general term of the series

$$A(1 + rx + r^2x^2 + r^3x^3 + \ldots\ldots + r^nx^n),$$

resulting from its expansion, is obviously Ar^nx^n; being so called, because, by substituting 0, 1, 2, 3, &c. successively for n, we shall obtain all the terms of the series.

Again, if the generating fraction has a polyno-

(*p*) When the series, whose sum is to be determined, is entirely numeral, as

$$1 + 2 + 8 + 28 + 100 + \&c.$$

its value must be found, as if it were of the form

$$1 + 2x + 8x^2 + 28x^3 + 100x^4 + \&c.$$

and then putting x in the resulting expression $= 1$.

mial for its denominator, which can be decomposed, according to the practical article, Vol. I, into simple fractions of the form

$$\frac{A}{1-rx},\quad \frac{A'}{1-r'x},\quad \frac{A''}{1-r''x},\ \&c.$$

the general term of the series resulting from the development of the proposed fraction, may be easily determined; for since

$$\begin{aligned}
&A\,(1+r\,x+r^2\,x^2+r^3\,x^3+\ \ldots\ldots\ +r^n\,x^n)\\
&A'\,(1+r'\,x+r'^2\,x^2+r'^3\,x^3+\ \ldots\ldots\ +r'^n\,x^n)\\
&A''(1+r''x+r''^2x^2+r''^3x^3+\ \ldots\ldots\ +r''^nx^n)\\
&\qquad\qquad \&c.
\end{aligned}$$

are the recurring series, which arise from the expansion of these simple fractions, and

$$A+Bx+Cx^2+Dx^3+\ \ldots\ldots\ +Rx^n+\&c.$$

is the development of the proposed fraction, we shall have, by hypothesis, the identical equation

$$\frac{a+bx+cx^2+dx^3+\ \ldots\ldots\ +px^{n-1}}{1-a'x-b'x^2-\ \ldots\ldots\ -qx^n}=$$
$$\frac{A}{1-rx}+\frac{A'}{1-r'x}+\frac{A''}{1-r''x}+\&c.$$

From which, by multiplication, and collecting the similar terms, there will arise

$$\left.\begin{array}{l} A\ \ +Ar \\ A'\ +A'r' \\ A''+A''r'' \end{array}\right|\ x+\ \ldots\ldots\ \left.\begin{array}{l} +Ar^n \\ +A'r'^n \\ +A''r''^n \end{array}\right|\ x^n+\&c.$$
$$=A+Bx+Cx^2+Dx^3+\ \ldots\ldots\ +Rx^n+\&c.$$

And, consequently, by comparing the coefficients of x^n, in each of these two series, we shall have

$$R=Ar^n+A'r'^n+A''r''^n+\&c.;$$

where it appears, that the coefficient of any term

x^n, of the recurring series, is equal to the sum of the coefficients of the same powers of x, which arise from expanding the several simple fractions into which the given fraction is decomposed.

Thus, as an example of this kind, let there be taken the rational fraction

$$\frac{1-x}{1-5x+6x^2}=\frac{-1}{1-2x}+\frac{2}{1-3x}.$$

Then, if the two simple fractions, into which the proposed fraction is decomposed, be converted into series, we shall have

$$\frac{-1}{1-2x}=-1-2x-4x^2-8x^3-\ldots\ldots-2^nx^n$$

$$\frac{2}{1-3x}=2+6x+18x^2+54x^3+\ldots\ldots+2.3^nx^n$$

where it is evident, that the general term of the series resulting from the generating fraction is

$$2.3^nx^n-2^nx^n, \text{ or } (2.3^n-2^n)x^n.$$

And, consequently, if 0, 1, 2, 3, &c. be successively substituted for n, the recurring series itself will be

$$1+4x+14x^2+46x^3+146x^4+\ldots+(2.3^n-2^n)x^n.$$

It may be observed, however, that this method of proceeding is indirect, and, in many cases, extremely laborious; in which instances it will be better to find the general term of the series by means of the multinomial theorem, Art. (A), p. 206, Vol. I; which may be readily done by putting the fraction

$$\frac{a+bx+cx^2+dx^3+\ldots\ldots+px^{m-1}}{1-a'x-bx^2-\ldots\ldots-qx^n}$$

under the following equivalent form,

$$(a+bx \ldots px^{m-1})(1-a'x-bx^{2} \ldots px^{m-1}-qx^{m})^{-1},$$

and then multiplying the expansion of the second member of this expression by the first.

We might here also proceed to show the truth of the rule, given in the note to the corresponding article, Vol. I, for determining whether any proposed series be recurrent or not; but this and other details belonging to the subject, cannot be entered upon without exceeding the limits of the present performance (*q*).

OF THE SUMMATION OF INFINITE SERIES.

(L) P. 325. THE general doctrine of series, being one of the most curious and interesting branches of the modern analysis, it may not be improper, in addition to what has been said upon this subject in the practical article (L), p. 325, of the former part of the work, to present the reader, in this place, with a slight sketch of some of the easiest and most useful methods, which have been employed by different authors, for obtaining the summation of such series of this kind, as are more immediately capable of an algebraical investigation; the rest being either connected with, or dependent upon principles, which belong to the higher departments of science.

(*q*) Besides the works referred to for farther information on this subject, in the practical article (K), p. 320, Vol. I, see a very ingenious memoir of Lagrange, entitled *Recherches sur la Maniére de former des Tables des Planètes, d'après les seules Observations*, an. 1772 of the Academy of Sciences of Paris, Part I.

With this view, therefore, it may be observed, as a matter chiefly of historical curiosity, that the celebrated Archimedes, to whose genius we are indebted for so many other improvements and discoveries in the mathematical sciences, appears to have been the first who determined the sum of all the terms of an infinite series of squares, whose roots are in arithmetical progression, and the common difference equal to the first term; which series he employed as one of the means of squaring the conical parabola, and for other purposes.

From that period, however, few traces are to be found of this subject, until, after a lapse of near two thousand years, Wallis, in his *Arithmetica Infinitorum*, published in 1655, first led the way to farther discoveries of this kind; which was, soon afterwards, prosecuted with far greater success, by Newton, Brounker, Mercator, and James Gregory; all of whom greatly enriched this part of the science, by the invention of several general methods for the summation and interpolation of series, that had never before been considered (*r*).

(*r*) It may here be remarked, that Wallis, in his *Opera Arithmetica*, published in 1657, first reduced the fraction $\frac{a}{1-r}$, by a perpetual division, into the infinite series

$$a + ar + ar^2 + ar^3 + ar^4 + \&c.$$

which conversion, though now thought of little importance, may be regarded, for the time in which it took place, as a very considerable step in analysis; having been the germ of many important improvements in that science.

For the dicoveries, in this branch, of the other writers mentioned above, see Newton's Method of Fluxions and Infinite

Leibnitz, also, in a paper entitled *De proportione Circuli ad quadratum circumscriptum in Numeris rationalibus*, published in the Leipsic Acts, 1682, gave several curious numerical series, of a different nature from the former; among which may be reckoned the following:

$$\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{35}+\frac{1}{48}+\&c.$$

Or

$$\frac{1}{2^2-1}+\frac{1}{3^2-1}+\frac{1}{4^2-1}+\frac{1}{5^2-1}+\&c.$$

The sum of an infinite number of terms of which is $\frac{3}{4}$.

Also, if the odd terms only of this series be taken, as

$$\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{63}+\frac{1}{99}+\&c.$$

the sum of the terms, when they are continued to infinity, will be $\frac{1}{2}$.

And the sum of the even terms of the same series, as

$$\frac{1}{8}+\frac{1}{24}+\frac{1}{48}+\frac{1}{80}+\frac{1}{120}+\&c.$$

taken as in the former instance, will be $\frac{1}{4}$.

Series, translated and published by Colson, in 1736; and his Analysis by Equations of an Infinite Number of Terms, and his Quadratures, both of which were translated and published by Stewart, in 1745. Also the well known treatise of James Gregory, entitled *Vera Circuli et Hyperbola quadratura*, printed at Padua in 1667, and reprinted the following year at Venice; and the *Logarithmotechnia* of Mercator, London, 1668.

Lastly, the sum of an infinite number of terms of the same series, omitting every three terms from the first to the fifth, from the fifth to the ninth, &c.; as

$$\frac{1}{3}+\frac{1}{35}+\frac{1}{99}+\frac{1}{195}+\frac{1}{323}+\&c.$$

Or

$$\frac{1}{2^2-1}+\frac{1}{6^2-1}+\frac{1}{10^2-1}+\frac{1}{14^2-1}+\frac{1}{18^2-1}+\&c.$$

will be equal to the area of the circle, of which the inscribed square is $\frac{1}{4}$; or that of the circumscribed square $\frac{1}{2}$.

But if we begin from the second term of the same series, and omit the three intermediate terms between that and the sixth, &c.; as

$$\frac{1}{8}+\frac{1}{48}+\frac{1}{120}+\frac{1}{224}+\frac{1}{360}+\&c.$$

Or

$$\frac{1}{3^2-1}+\frac{1}{7^2-1}+\frac{1}{11^2-1}+\frac{1}{15^2-1}+\frac{1}{19^2-1}+\&c.$$

The sum of an infinite number of these terms will be equal to the area of the space included between the curve and the asymptotes of an equilateral hyperbola, or $\frac{1}{4}$ hyp. log. 2.

He also gave, in 1683, in the same work, the summation of several other series, of a more difficult kind, which have the numerators of the different terms either unity, or the natural, or triangular numbers, &c. and their denominators, certain numbers in geometrical progression, the signs being alternately + and −; as

$$1-\frac{1}{2.10}+\frac{1}{2.10^2}-\frac{1}{2.10^3}+\frac{1}{2.10^4}-\&c.$$

$$1-\frac{2}{2.10}+\frac{3}{2.10^2}-\frac{4}{2.10^3}+\frac{5}{2.10^4}-\&c.$$

$$1-\frac{3}{2.10}+\frac{6}{2.10^2}-\frac{10}{2.10^3}+\frac{15}{2.10^4}-\&c.$$

The sum of an infinite number of terms of which are, respectively,

$$\frac{20}{21},\ \frac{400}{441},\ \text{and}\ \frac{8000}{9261}.$$

Leibnitz, indeed, barely announced these series in the publication above mentioned; but the truth of them was soon after demonstrated, in different ways, by James and John Bernoulli; the former of whom also gave, in a small tract entitled *Positiones Arith. de Serieb. Infinitis*, &c. 1692 and 1697, various other series, of the form,

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\&c.$$

$$\frac{1}{2}+\frac{3}{4}+\frac{6}{8}+\frac{10}{16}+\frac{15}{32}+\&c.$$

$$\frac{1}{2}+\frac{6}{4}+\frac{10}{8}+\frac{20}{16}+\frac{35}{32}+\&c.$$

the sums of an infinite number of terms of which are respectively equal to 2, 4, and 8.

He here likewise shows that the sum of the reciprocals of the series of triangular numbers

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\&c.=2.$$

And that the sum of the reciprocals of the following polygonal numbers,

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}+\&c.=1\tfrac{1}{2}.$$

Which, though this branch of the subject has

since been greatly extended, and rendered more general, was a considerable step for that time.

One of the means used by the two illustrious brothers above mentioned, in the resolution of problems of this kind, was to convert the proposed series into several others whose summation is known; but that which they more frequently employed, consists in subtracting from an assumed series the same series deprived of some of its leading terms; in which case, the sum of the resulting series will be determined.

Thus, supposing the following series of the reciprocals of the natural numbers,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\&c.=s,$$

without regarding whether the sum be finite or infinite (*s*).

Then, if from this sum, or the value of s, there be taken the same series, deprived of its first term, we shall have,

$$\begin{array}{l} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\&c.=s. \\ \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\&c.=s-1. \\ \hline \end{array}$$

$$\text{Or } \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}+\&c.=1.$$

(*s*) The sum of this series, although its terms continually decrease, can be readily shown to be infinite, or greater than any assignable number; as will be done in the next article, which treats of the theory of logarithms; under which head the demonstration properly falls.

Where it appears that this last series, continued to infinity, is equal to 1.

In like manner, if the two first terms of the same series be omitted, and the rest be subtracted from it, there will arise

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c. = s$$

$$\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \&c. = s - \frac{3}{2}$$

$$\frac{2}{1.3} + \frac{2}{2.4} + \frac{2}{3.5} + \frac{2}{4.6} + \frac{2}{5.7} + \&c. = \frac{3}{2}$$

Or

$$\frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} + \frac{1}{4.6} + \frac{1}{5.7} + \&c. = \frac{3}{4}$$

Which, as before observed, is the sum of this last series, continued ad infinitum.

Also, if the first term of this last series be omitted, and the rest subtracted from it, we shall have

$$\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{20} + \frac{1}{35} + \&c. = \frac{3}{4}$$

$$\frac{1}{8} + \frac{1}{15} + \frac{1}{20} + \frac{1}{35} + \frac{1}{48} + \&c. = \frac{5}{12}$$

Or $\frac{5}{2.3.4} + \frac{7}{2.3.4.5} + \frac{9}{2.3.4.5.6} + \frac{11}{2.3.4.5.6.7} + \&c. = \frac{1}{3}$

Where the law of the terms, both in the numerator and denominator, is sufficiently obvious.

The same results may also be readily obtained by the method made use of by de Moivre (*Miscel. Analyt.* lib. vi, cap. 3); which consists in multiplying an assumed series by some binomial or trinomial expression, and then equating that ex-

pression to 0, and transposing the negative terms.

Thus, let there be taken, as in the former instances, the series

$$1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\frac{1}{5}x^4\&c.=s.$$

Then, if this be multiplied by the binomial $x-1$, we shall have

$$-1+\frac{1}{1.2}x+\frac{1}{2.3}x^2+\frac{1}{3.4}x^3+\frac{1}{4.5}x^4\&c.=(x-1)s.$$

Whence, taking $x-1=0$, or $x=1$, and transposing the negative term -1, there will arise

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}\&c.=1.$$

Which is, therefore, the sum of all the terms of this last series, continued to infinity.

Again, if we take, as a second example of this kind, the same series

$$1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\frac{1}{5}x^4\&c.=s.$$

This, multiplied by the binomial x^2-1, will give

$$-1-\frac{1}{2}x+\frac{2}{1.3}x^2+\frac{2}{2.4}x^3+\frac{2}{3.5}x^4+\&c.=(x^2-1)s.$$

Where, putting $x=1$, and transposing the two first terms, we shall have

$$\frac{2}{1.3}+\frac{2}{2.4}+\frac{2}{3.5}+\frac{2}{4.6}+\frac{2}{5.7}\&c.=\frac{3}{2}.$$

Which are the same as the sums before found by the method of the Bernoullis.

And if, for a third example, there be again taken the same series,

$$1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\frac{1}{5}x^4\&c.=s,$$

This, multiplied by $(2x-1)(3x-1)$, or by the trinomial $6x^2-5x+1$, will give

$$1-\frac{9}{2}x+\frac{23}{1.2.3}x^2+\frac{38}{2.3.4}x^3+\frac{57}{3.4.5}x^4\&c.=(2x-1)(3x-1)s$$

Where, putting $2x-1=0$, or $3x-1=0$, which gives $x=\frac{1}{2}$, or $x=\frac{1}{3}$, we shall have, by transposing $1-\frac{9}{2}x$, the two following series:

$$\frac{23}{1.2.3}\left(\frac{1}{4}\right)+\frac{38}{2.3.4}\left(\frac{1}{8}\right)+\frac{57}{3.4.5}\left(\frac{1}{16}\right)+\frac{80}{4.5.6}\left(\frac{1}{32}\right)\&c.=\frac{5}{4}$$

$$\frac{23}{1.2.3}\left(\frac{1}{9}\right)+\frac{38}{2.3.4}\left(\frac{1}{27}\right)+\frac{57}{3.4.5}\left(\frac{1}{81}\right)+\frac{80}{4.5.6}\left(\frac{1}{243}\right)\&c.=\frac{1}{2}$$

The differences of the numerators of each of which are in arithmetical progression.

And, in a manner similar to the mode of proceeding here followed, the sums of a great variety of other infinite series may be readily found.

These methods, however, are chiefly applicable to such series as have determinate values; but for those of the form

$$a-bx+cx^2-dx^3+ex^4\&c.$$

which converge slowly, and that cannot be exactly summed, the following differential series,

$$a-b\left(\frac{x}{1+x}\right)-D'\left(\frac{x}{1+x}\right)^2-D''\left(\frac{x}{1+x}\right)^3-D'''\left(\frac{x}{1+x}\right)^4\&c.\ (t)$$

which is eqvivalent to the former, may be used, to

(t) The proposed series, in this mode of solution, may be readily converted into the differential series, by assuming

$$a-bx+cx^2-dx^3\&c.=a+p\left(\frac{x}{1+x}\right)+q\left(\frac{x}{1+x}\right)^2+r\left(\frac{x}{1+x}\right)^3\ \&c.$$

and then comparing the coefficients of the several powers of x in the latter, when its different terms are developed, and brought properly together, with the like powers of x in the former.

great advantage in determining more readily their approximate values; observing that

$D' = b - c$, $D'' = b - 2c + d$, $D''' = b - 3c + 3d - e$, &c.

Thus, if it were required to find the approximate sum of the series

$$x - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \frac{1}{5}x^5 - \&c.$$

which is well known to denote the hyperbolic, or Neperian logarithm of $1 + x$; the above differential series, in this case, will become,

$$a\left(\frac{x}{1+x}\right) + D'\left(\frac{x}{1+x}\right)^2 + D''\left(\frac{x}{1+x}\right)^3 + D'''\left(\frac{x}{1+x}\right)^4 \&c.$$

Where $a = 1$, $b = \frac{1}{2}$, $c = \frac{1}{3}$, $d = \frac{1}{4}$, &c.

And $D' = \frac{1}{2}$, $D'' = \frac{1}{3}$, $D''' = \frac{1}{4}$, $D^{iv} = \frac{1}{5}$, &c.

Whence

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \&c. = \frac{x}{1+x} + \frac{1}{2}\left(\frac{x}{1+x}\right)^2 + \frac{1}{3}\left(\frac{x}{1+x}\right)^3 \&c.$$

And if x, in each of these two series, be now taken $= 1$, we shall have

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \&c. = \frac{1}{2} + \frac{1}{2.4} + \frac{1}{3.8} + \frac{1}{4.16} + \frac{1}{5.32} \&c.$$

Which latter series converges so much faster than the former, that the sum of 25 of its terms will give the approximate value sought more exactly than the sum of 10000 terms of the other.

Again, suppose it were required to find a near value of the series

$$1 - \frac{1}{3}x + \frac{1}{5}x^2 - \frac{1}{7}x^3 + \frac{1}{9}x^4 - \frac{1}{11}x^5 + \&c.$$

Which, when $x = 1$, denotes $\frac{1}{8}$th of the circumference of the circle, whose radius is 1.

In this case, we shall have, by using the differ-

ential series in its first form, and following the same process as above,

$$1-\frac{1}{3}x+\frac{1}{5}x^2-\frac{1}{7}x^3+\frac{1}{9}x^4-\&c.=1-\frac{1}{3}\left(\frac{x}{1+x}\right)-\frac{2}{3.5}\left(\frac{x}{1+x}\right)^2$$
$$-\frac{2.4}{3.5.7}\left(\frac{x}{1+x}\right)^3-\frac{2.4.6}{3.5.7.9}\left(\frac{x}{1+x}\right)^4-\frac{2.4.6.8}{3.5.7.9.11}\left(\frac{x}{1+x}\right)^5\&c.$$

And, consequently, by taking $x=1$, there will arise the numeral series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\&c.=1-\frac{1}{3}\left(\frac{1}{2}\right)-\frac{2}{3.5}\left(\frac{1}{4}\right)-\frac{2.4}{3.5.7}\left(\frac{1}{8}\right)-$$
$$\frac{2.4.6}{3.5.7.9}\left(\frac{1}{16}\right)-\frac{2.4.6.8}{3.5.7.9.11}\left(\frac{1}{32}\right)-\&c.$$

Which last converges much faster than $1-\frac{1}{3}+\frac{1}{5}-$ &c.; but it is not so rapid as that before given for the hyp. log. of 2.

When any slowly converging series has all its terms positive, or is of the form

$$a+bx+cx^2+dx^3+ex^4+\&c.$$

neither of the two following differential series

$$a-\frac{bx}{1+x}-D'\left(\frac{x}{1+x}\right)^2-D''\left(\frac{x}{1+x}\right)^3-D'''\left(\frac{x}{1+x}\right)^4-\&c.$$

Nor,

$$a+\frac{bx}{1-x}-D'\left(\frac{x}{1-x}\right)^2+D''\left(\frac{x}{1-x}\right)^3-D'''\left(\frac{x}{1-x}\right)^4+\&c.$$

can, in this case, be advantageously employed for rendering its convergency more rapid; as the several powers of x, and their coefficients, will not decrease together, or at the same time.

Hence, for the summation of any series of this kind, recourse must be had to some of the methods that have been devised by different writers for this purpose, most of which are too complicated and la-

borious to be inserted in a work like the present (*v*).

A similar reference must, also, be made for the methods of solution given by Euler and John Bernoulli, for summing a series of the form

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\&c.$$

Of which, even in the most simple case, where $n=2$, the difficulty was considered, by James Bernoulli, as too great to be surmounted.

John Bernoulli, however, after the death of his brother, obtained, by means of resolving the common expression for the sine of a circular arc (*Opera*, tom. iv) into binomial factors, the solution of the case last mentioned; and afterwards extended the method he had made use of to various other series of the form

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\&c.$$

Showing, when n is any even number, that

(*v*) It is a remarkable circumstance, as has long since been observed by Stirling, in his learned Treatise on the Summation of Infinite Series, that a slowly converging series of the form

$$a+bx+cx^2+dx^3+ex^4+\&c.$$

which has all its terms positive, should be found so much more difficult to be summed, to any considerable degree of exactness, than a similar series of the form

$$a-bx+cx^2-dx^3+ex^4-\&c.$$

where the signs are + and − alternately; which latter, even in the most difficult cases, can always have its sum ascertained by means of the differential theorem, to eight or nine places of figures, by the mere computation of a few of its first terms, while that of the former can only be obtained, to the same extent, by the most operose and tedious process.

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\&c.=\frac{\pi^2}{6}$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\&c.=\frac{\pi^4}{90}$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\&c.=\frac{\pi^6}{945}$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\&c.=\frac{\pi^8}{9450}$$

&c.

Where π denotes the semicircumference of the circle to radius 1.

But those cases, in which n is an odd number, not falling under this process, the subject was resumed by Euler, who gave a method of approximating to the values of such series of this kind as could not be definitively expressed by circular arcs (*u*).

Euler also converted the following series, first given by Wallis,

$$\frac{1.3}{2^2}\times\frac{3.5}{4^2}\times\frac{5.7}{6^2}\times\frac{7.9}{8^2}\times\frac{9.11}{10^2}\&c.$$

for expressing the ratio of the circle to the square of its diameter, into a series of the common form; from which he derived a number of curious consequences.

(*u*) For the mode by which Euler obtained the sums of the series here mentioned, as well as of many others of a similar nature, see his *Introductio in Analysin Infinitorum*, where nearly every branch of the transcendental analysis is investigated and exemplified in the most luminous and masterly manner.

Neither Euler nor Bernoulli, however, were able to obtain the several values of the series

$$\pm x-\frac{x^2}{2^n}\pm\frac{x^3}{3^n}-\frac{x^4}{4^n}\pm\frac{x^5}{5^n}\&c.$$

except in the particular case of $x=1$; but this part of the subject has lately been reconsidered, and ably treated on, by

He likewise showed that the hypergeometrical series

$$1 - 1 + 2 - 6 + 24 - 120 + \&c.$$

which, considering the rapidity with which its terms increase, would be judged to be infinite, has, for its approximate sum, .596347362 1237.

To this we may farther add, that Monmort, in his *Essay d'Analyse sur les Jeux de Hazard*, p. 65, 2d edit. 1713, first converted the series

$$a,\ b,\ c,\ d,\ e,\ f,\ g,\ \&c.$$

into another of the following equivalent form

$$na + \frac{n(n-1)}{1.2}D' + \frac{n(n-1)(n-2)}{1.2.3}D'' + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}D''' \&c.$$

where n denotes the number of terms, and D', D'', D''', &c. the first, second, third, &c. differences, as before.

Which latter expression is such, that if any of these differences become $= 0$, the series will terminate at that point, and give the sum of all the preceding terms.

Thus, taking, for example, the following series of polygonal numbers,

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Mr. Spence, in his *Essay on the Theory of the various Orders of Logarithmic Transcendents*, 1809.

Landen, also, in his *Lucubrations* and *Mathematical Memoirs*, 1755 and 1780, has given the investigations of various series of this and other kinds, that had not hitherto been considered with the attention they deserved; particularly the series

$$1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} \&c.$$

The sum of which he has obtained in a direct and perspicuous manner.

$$1+4+10+20+35+56+\&c.$$

we shall have, for the sum of the first 1000 terms,

$$na+\frac{n(n-1)}{1.2}D'+0=1000+499500=500500\,(x).$$

Various other interesting particulars, relating to this subject, might also be here given; particularly the methods made use of for finding the general term of a series; and those for determining whether the sum of any proposed series be finite or infinite: but any detail of this kind would exceed the limits of the present performance.

The reader, therefore, who may wish to see these and other circumstances of this kind more fully explained, may consult the learned work of Stirling, before mentioned, and Maclaurin's Fluxions; in the latter of which performances, he will find an investigation of the last of the abovementioned particulars, from which may be derived the following rule:

Take any three equidistant terms, p, q, r, of a given series; then, if

$$\frac{p(q-r)}{p-q}=r,$$

it may be concluded, that the sum of the series in question is finite.

(x) See, on this subject, and other branches of the theory of series, and the doctrine of probabilities, the correspondence that passed between Montmort, John Bernoulli, and his nephew Nicolas, given at the latter end of the second edition of *Les Jeux de Hazard*, above mentioned; which contains a curious history of the early attempts that were made to investigate this branch of the science, and may be considered as a model of the manner in which disputes of this kind should be conducted.

For the summation, likewise, of a great variety of series, which do not fall under any of the preceding rules, the learner may consult Simpson's Miscellaneous Tracts, where he will find a number of theorems, for this purpose, which are, perhaps, more general in their application than those of any other writer.

THEORY OF LOGARITHMS.

(M) P. 357. In addition to what has been said, on the subject of the present article, in the former part of the work, it may here be farther observed, that the series now commonly used, in the computation of logarithms, were originally derived from the hyperbola; by means of which, as well as by the logistic curve, the nature and properties of these numbers can be clearly and elegantly explained.

The doctrine, however, being purely algebraical, this mode of illustrating it, by the intervention of certain curves, was justly considered by Halley as not conformable to the nature of the subject; which manifestly requires, that its truth should be made to rest solely upon deductions, derived from the rules and principles of that science.

He accordingly published a paper, on this subject, in the Philosophical Transactions for 1695, in which he attempted to investigate the same series from the abstract principles of numbers; but the method he has followed, which is a kind of disguised fluxions, is, in many places, so extremely abstruse and obscure, that few have been able to comprehend his reasoning.

As the theory of this important branch of the science was, therefore, still left in an imperfect state, the author of the present performance was induced to furnish a more natural and perspicuous method of investigating these series; which was published in the Supplement to Hutton's Mathematical Dictionary, 1795; and is deduced solely from algebraical principles, without any reference to the doctrine of curves.

And as the mode of proceeding there used, though in some respects different from others that have since been employed, appears to be better adapted to the comprehension of learners than any I have yet seen, I shall here endeavour, by means of a few slight alterations and improvements, to render it as clear and satisfactory as possible.

For this purpose, let there be taken, as in the practical article, Vol. I, p. 357,

$$r^x = n.$$

Then, as is there observed, the logarithm of n is x; which may be either positive or negative, and r any number whatever, according to the different systems of logarithms that may arise.

When $n=1$, it is evident that x must be $=0$, whatever may be the value of r; and, consequently, the logarithm of 1 is always 0 in every system of logarithms.

Also, when $x=1$, it is manifest that n must be $=r$; which last symbol is usually called the radix, or base; being that number, in every system, whose logarithm, in that system, is 1.

Hence, to find the logarithm of any number, in

any system of logarithms, it is only necessary, from the equation

$$r^x = n,$$

to determine the value of the index x, in terms of the base r and the given number n; which may be done as follows.

Let there be taken, instead of the proposed exponential, the equivalent expression

$$r^x = \{1 + (r-1)\}^x.$$

Then, by expanding this last formula, after the manner of the binomial theorem, Art. (Y), we shall have

$$\{1 + (r-1)\}^x =$$
$$1 + x(r-1) + \frac{x(x-1)}{1 \cdot 2}(r-1)^2 + \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3}(r-1)^3 \&c.$$

And if the right hand member of this equation be arranged according to the several powers of x, the result will give

$$r^x = 1 +$$
$$\{(r-1) - \frac{1}{2}(r-1)^2 + \frac{1}{3}(r-1)^3 - \frac{1}{4}(r-1)^4 + \&c.\}x +$$
$$\{\frac{1}{2}(r-1)^2 - \frac{1}{2}(r-1)^3 + \&c.\}x^2 +$$
$$\{\frac{1}{6}(r-1)^3 - \&c.\}x^3 + \&c.$$

From which it is evident, that the exponential $\{1 + (r-1)\}^x$, or its equal r^x, may be put under the form of the following series (y):

(y) It may here be observed, that if the series in this, or any other similar case, be assumed of an improper form, either the function from which the development is supposed to be derived will not, when x is put $= 0$, agree with that development, or the comparison of the coefficients of its different terms, with those of some other expansion of it, will lead to contradictory equations.

$$r^x = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 \&c. \ldots (1)$$

And, consequently, since the coefficients A, B, C, &c. in this expression, are independent of any particular value of the index x, we shall also have

$$r^{2x} = 1 + A(2x) + B(2^2x^2) + C(2^3x^3) + D(2^4x^4) \&c.$$

Or

$$r^{2x} = 1 + 2Ax + 4Bx^2 + 8Cx^3 + 16Dx^4 + \&c. \ldots (2)$$

Whence, by squaring the right hand member of equation (1) and putting it equal to the right hand member of equation (2), there will arise

$$\begin{array}{r|r|r|r|l} 1 + 2Ax + 2B & x^2 + 2C & x^3 + 2D & x^4 + 2E & x^5 + \&c. \\ + A^2 & + 2AB & + 2AC & + 2AD & \\ & & + B^2 & + 2BC & \end{array}$$

$$= 1 + 2Ax + 4Bx^2 + 8Cx^3 + 16Dx^4 + 32Ex^5 + \&c.$$

And, therefore, by comparing the coefficients of the homologous terms of these two formulæ, we shall have

$$B = \frac{1}{2}A^2$$

$$C = \frac{1}{2 \cdot 3}A^3$$

$$D = \frac{1}{2 \cdot 3 \cdot 4}A^4$$

$$E = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}A^5$$

&c.

Whence

$$1 + Ax + \frac{A^2}{2}x^2 + \frac{A^3}{2 \cdot 3}x^3 + \frac{A^4}{2 \cdot 3 \cdot 4}x^4 + \&c. = r^x = n.$$

Or

$$Ax + \frac{A^2}{2}x^2 + \frac{A^3}{2.3}x^3 + \frac{A^4}{2.3.4}x^4 + \frac{A^5}{2.3.4.5}x^5 + \&c. = n - 1.$$

And, consequently, by reverting this last series, according to the method pointed out in Art. (B), p. 185, we shall have

$$x = \frac{(n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \frac{1}{4}(n-1)^4 + \&c.}{A}$$

Or, since, from what has been shown in the former part of the investigation,

$$A = (r-1) - \frac{1}{2}(r-1)^2 + \frac{1}{3}(r-1)^3 - \frac{1}{4}(r-1)^4 + \&c.$$

if this value be substituted for A in the denominator of the right hand member of the last equation, there will arise

$$x = \frac{(n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \frac{1}{4}(n-1)^4 + \&c.}{(r-1) - \frac{1}{2}(r-1)^2 + \frac{1}{3}(r-1)^3 - \frac{1}{4}(r-1)^4 + \&c.}$$

which is, therefore, a general expression for the logarithm of any number n, in any system of logarithms; the radix, or base, r, being taken of any value, greater or less than 1.

But as r, in every system, is a constant quantity; being always the number whose logarithm in the system to which it belongs is 1, the above expression may be simplified, either by assuming $r =$ to some particular number, and thence finding the value of the series constituting the denominator; or by assuming this whole series $=$ to some particular number, and thence finding the value that must be given to the radix r.

By the latter of these methods, the denominator may be made to vanish, or become of no effect, by

assuming the value of the series of which it consists $=1$; in which case x, or

$$\log.n=(n-1)-\frac{1}{2}(n-1)^2+\frac{1}{3}(n-1)^3-\frac{1}{4}(n-1)^4+\&c.$$

And, by reverting the abovementioned series,

$$(r-1)-\frac{1}{2}(r-1)^2+\frac{1}{3}(r-1)^3-\frac{1}{4}(r-1)^4+\&c.=1,$$

according to the method before referred to, we shall have, for this particular case,

$$r=2.718281828459\&c.$$

Where it may be observed, that the system arising from this mode of determining the radix r, is that which furnishes what have been usually called hyperbolic, or Neperian logarithms; and appears to be the simplest that can be formed (z).

If, on the contrary, the radix r be assumed = to some particular number, as, for instance, 10, we shall have, in this case,

$$(r-1)-\frac{1}{2}(r-1)^2+\frac{1}{3}(r-1)^3-\frac{1}{4}(r-1)^4+\&c.=$$
$$2.30258509\&c.$$

$$\text{and } \log.n=\frac{1}{2.302585\&c.}\times$$
$$\left\{(n-1)-\frac{1}{2}(n-1)^2+\frac{1}{3}(n-1)^3-\frac{1}{4}(n-1)^4+\&c.\right\}$$

which last expression gives the system that furnishes Briggs's, or the common tabular logarithms.

(z) The term hyperbolic, as applied to the species of logarithms here mentioned, should be wholly dropt, in treating of this subject, and the name of the inventor used exclusively in its stead; as the appellation is now well known to be improper; since every system of logarithms may be derived from the properties of the hyperbola; as was first clearly shown by Maseres, in his Elements of Plane Trigonometry, published in 1760.

And, in the same manner, by assuming any particular value for r, and thence determining the value of the series

$$(r-1)-\frac{1}{2}(r-1)^2+\frac{1}{3}(r-1)^3-\frac{1}{4}(r-1)^4+\&c.$$

Or, by assuming the same series of some particular value, and thence determining the value of r, any system of logarithms may be derived.

The number A, above used, or its equal, the series

$$(r-1)-\frac{1}{2}(r-1)^2+\frac{1}{3}(r-1)^3-\frac{1}{4}(r-1)^4+\&c.$$

is similar to that which was first called by Cotes, in his *Harmonia Mensurarum*, 1722, the modulus of the system; being always a constant quantity, depending on the assumed value of the radix r.

And as the form of this series is exactly the same as that which constitutes the numerator; which has been shown to be the Neperian logarithm of n, it follows, that the modulus of any system of logarithms is equal to the Neperian logarithm of the radix of that system; or of the number whose proper logarithm, in the system to which it belongs, is 1.

Hence, taking the term according to the acceptation here mentioned, we shall have

$$\text{A}=lr, \text{ and } \frac{1}{\text{A}}, \text{ or } \text{M}=\frac{1}{lr};$$

where l denotes the Neperian logarithm of the radix r, and M the reciprocal of the modulus.

So that, when $r=2.718281828459$&c. as in the system of Neperian logarithms,

$$\text{A, or } lr = 1, \text{ and M, or } \frac{1}{lr} = 1.$$

And when $r = 10$, as in the common system of logarithms,

$$\text{A, or } lr = 2.30258509, \text{ and}$$

$$\text{M, or } \frac{1}{lr} = .434294819, \quad . \quad . \quad (a).$$

Hence, $\text{M} \times \text{Nep. log. } n = \text{Com. log. } n$, and $\frac{1}{\text{M}} \times \text{Com. log. } n, = \text{Nep. log. } n$.

The form of the series, above obtained, for the Neperian logarithm of n, which, by substituting $1 + n$ for n, will become

$$\log. (1 + n) = n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \&c.$$

is the same as that which was first given by Mercator in his *Logarithmotechnia*, and by James Gregory, in his *Exercitationes Geometricæ*; both of which works were published in 1668.

Gregory, also, in the performance here mentioned, gave another series of this kind; which is equivalent to

$$\log. (1 - n) = -n - \frac{1}{2}n^2 - \frac{1}{3}n^3 - \frac{1}{4}n^4 - \&c.$$

And Wallis, who, in the same year, published a paper on this subject, in the Philosophical Transactions, gave the series

$$\log. \frac{1 + n}{1 - n} = 2(n + \frac{1}{3}n^3 + \frac{1}{5}n^5 + \frac{1}{7}n^7 + \&c.)$$

(*a*) The modulus, or lr, instead of being found from the denominator of the series above investigated, which cannot be used when $n =$ or $\asymp 1$, may be obtained from the third series in the next page, which is far more convergent; and by others hereafter given; observing to put r for a, and $\text{M} = 1$.

the former of which is readily obtained from that above given, by putting $-n$ for n; and the latter, by subtracting the second of these from the first.

Hence, taking $\frac{1}{lr}$, or M, for the modulus, as before shown, the logarithm of any number n may be exhibited universally, or according to any system of logarithms, in the three following forms:

$$\log.(1+n) = \text{M}\{n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 + \&c.\}$$

$$\log.\ \frac{1}{1-n} = \text{M}\{n + \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 + \tfrac{1}{4}n^4 + \&c.\}$$

$$\log.\ \frac{1+n}{1-n} = 2\text{M}\{n + \tfrac{1}{3}n^3 + \tfrac{1}{5}n^5 + \tfrac{1}{7}n^7 + \&c.\}$$

Or

$$\log.\ n =$$

$$\text{M}\{(n-1) - \tfrac{1}{2}(n-1)^2 + \tfrac{1}{3}(n-1)^3 - \tfrac{1}{4}(n-1)^4 + \&c.\}$$

$$\log.\ n =$$

$$\text{M}\{(\tfrac{n-1}{n}) + \tfrac{1}{2}(\tfrac{n-1}{n})^2 + \tfrac{1}{3}(\tfrac{n-1}{n})^3 + \tfrac{1}{4}(\tfrac{n-1}{n})^4 + \&c.\}$$

$$\log.\ n =$$

$$2\text{M}\{(\tfrac{n-1}{n+1}) + \tfrac{1}{3}(\tfrac{n-1}{n+1})^3 + \tfrac{1}{5}(\tfrac{n-1}{n+1})^5 + \tfrac{1}{7}(\tfrac{n-1}{n+1})^7 + \&c.\}$$

The first of which series will converge only when n differs but little from 1; and the second is not much superior in this respect to the first; but the third will converge when n is any number whatever; but more slowly as n becomes larger.

By the addition and subtraction of these series, others may also be derived, of a greater or less degree of convergency; but in the direct computation of the logarithm of any given number, they will

be found to possess little or no advantage over the former (*b*).

In the determination, however, of the logarithm of a number, by means of the logarithms of other numbers that are supposed to be known, various methods may be employed, which will greatly facilitate the calculation.

Thus, for instance, if there be put, in the series above mentioned,

$$\frac{1+n}{1-n} = 1 + \frac{u}{n};$$

which gives

$$n = \frac{u}{2n+u},$$

we shall have, by substituting these expressions for their equals in that series,

$$\log.(1+\frac{u}{n}) = 2\text{M}\{(\frac{u}{2n+u}) + \frac{1}{3}(\frac{u}{2n+u})^3 + \frac{1}{5}(\frac{u}{2n+u})^5 + \&c.\}$$

Or

$$\log.(n+u) =$$

$$\log.n + 2\text{M}\{(\frac{u}{2n+u}) + \frac{1}{3}(\frac{u}{2n+u})^3 + \frac{1}{5}(\frac{u}{2n+u})^5 + \&c.\}$$

Where, if u be now taken $=1$, the last of these series will become

(*b*) Arbogast, in his treatise *Du Calcul des Dérivations*, p. 239, has given a theorem, one of the uses of which he considers to be that of enabling us to convert any proposed series into another, that shall be more convergent; but this, as Woodhouse has justly remarked, in his Principles of Analytical Calculation, cannot be easily shown, as the coefficients may decrease while the arbitrary quantity increases, and *vice versa*; for which reason, it must appear extremely doubtful, whether a series can be found, from which the logarithm of any proposed number can be directly obtained.

$$\log.(n+1) =$$
$$\log.n + 2\text{M}\{(\frac{1}{2n+1}) + \frac{1}{3}(\frac{1}{2n+1})^3 + \frac{1}{5}(\frac{1}{2n+1})^5 + \&c.\}$$

which converges very rapidly; being the same as that given in Art. (M), Vol. I, p. 357, for computing the logarithm of any number, when the logarithm of the preceding number is known.

In like manner, if n, in the same series, be changed into u, and there be put

$$\frac{1+u}{1-u} = \frac{n^2}{n^2-1};$$

which gives

$$u = \frac{1}{2n^2-1},$$

we shall have, by substituting these expressions for their equals, as before,

$$\log.\frac{n^2}{n^2-1} = 2\text{M}\{\frac{1}{2n^2-1} + \frac{1}{3(2n^2-1)^3} + \frac{1}{5(2n^2-1)^5} + \&c.\}$$

Or

$$\log.n =$$
$$\frac{1}{2}\{\log.(n+1) + \log.(n-1)\} + \text{M}\{\frac{1}{2n^2-1} + \frac{1}{3(2n^2-1)^3} + \&c.\}$$

Which series, from its great convergency, may be used for finding the logarithms of all the prime numbers, above 1000, to twelve places of decimals, provided those of the prime numbers under 1000 are given to fifteen places of decimals.

For, if n be a number a little greater than 1000, the second term of the series, when computed numerically, will be found to have only an effective figure in the twenty-first place of decimals.

Whence, neglecting this and the following terms, as having no effect in the calculation, we shall have

$$\log. n = \frac{1}{2}\{\log. (n+1) + \log. (n-1)\} + \text{M}\left(\frac{1}{2n^2-1}\right).$$

And if, instead of $\frac{1}{2n^2-1}$, there be taken $\frac{1}{2n^2-2}$, the error, thus arising, will be expressed by their difference

$$\text{M}\left\{\frac{1}{(2n^2-1)\times(2n^2-2)}\right\};$$

which, in the most favourable case of this kind, amounts only to

.00000000000001.

So that, in order to have the logarithm of the proposed number exact to twelve or thirteen places of decimals, it will be sufficient to take

$$\log. n = \frac{1}{2}\{\log. (n+1) + \log. (n-1)\} + \text{M}\left(\frac{1}{2n^2-2}\right).$$

Or

$$\log. n = \frac{1}{2}\{\log. (n+1) + \log. (n-1)\} + \frac{\frac{1}{2}\text{M}}{(n+1)(n-1)}.$$

Where, since n is supposed to be a prime number, the factors $n+1$ and $n-1$ will be even numbers; and, consequently, can be each decomposed into others less than n.

Hence, if the logarithm of half the reciprocal of the modulus, or $\log.\text{M} - \log. 2$, be taken from the tables in common use, and the logarithms of the rest of the terms of the expression from a table of the first 1000 numbers, calculated to fifteen places of decimals, the result will fall within the limits above mentioned (c).

(c) Several other artifices might have been here used for transforming a series of this kind into others, that are more or

With respect to the three principal logarithmic series above given, it is evident, that the convergency of the second

$$\log. n =$$
$$M\{(\frac{n-1}{n}) + \frac{1}{2}(\frac{n-1}{n})^2 + \frac{1}{3}(\frac{n-1}{n})^3 + \frac{1}{4}(\frac{n-1}{n})^4 + \&c.\}$$

will diminish in proportion as the number n is greater; and, consequently, as n augments, the fraction $\frac{n-1}{n}$ will approach continually towards unity as its limit.

Whence, supposing n to be an indefinitely large number, we shall have, in this case;

$$\log. n = M\{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c.\}.$$

And, therefore, as the logarithm increases at the same time with the number to which it belongs, the value of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \&c.$$

when $M = 1$, must, as before observed, Art. (L), p. 238, be indefinite, or greater than any number that can be named.

And, if the same series be taken of the following form,

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less advantageous, according to the circumstances attending them; but what has been above shown will be sufficient to answer most common purposes.

Those who may wish to see some of the most ingenious of these methods, may consult my Treatise of Plane and Spherical Trigonometry, p. 344, second edition, where they will find, among others, the transformations of Borda and Haros, which are so often cited by foreign writers on this subject.

$$\log.(1-n) = -\text{M}\{n + \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 + \tfrac{1}{4}n^4 + \&c.\}$$

it is manifest, when n, in this expression, is taken $=1$, that

$$\log.0 = -\text{M}\{1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \&c.\}$$

From which it appears, that the logarithm of 0, in every system, is infinitely negative.

Or the same result may be derived immediately from the primitive equation

$$r^x = n.$$

For, by taking the logarithms of each of its two members, we shall have $x \log. r = \log. n$; or

$$x = \frac{\log. n}{\log. r}, \text{ or } r^{\frac{\log. n}{\log. r}} = n.$$

Where it is evident, that while $\log. n$ remains positive, however small it may be, n will be greater than unity; and if n be a fraction, in which case

$$r^{-\frac{\log. n}{\log. r}} = n, \text{ or } \frac{1}{r^{\frac{\log. n}{\log. r}}} = \frac{1}{n},$$

it is easy to perceive, that as the second of the two members becomes less, the greater must be the index of the first, in order that the equality may subsist.

Whence it follows, that as the number n approaches towards 0, the logarithm of n will approach continually towards infinity, as was to be shown.

The first of the properties abovementioned may likewise be shown, in a similar manner, to belong to the series

$$\log. n = 2\text{M}\{(\tfrac{n-1}{n+1}) + \tfrac{1}{3}(\tfrac{n-1}{n+1})^3 + \tfrac{1}{5}(\tfrac{n-1}{n+1})^5 + \&c.\}$$

the convergency of which evidently diminishes like the former, in proportion as n becomes greater.

And as the fraction $\frac{n-1}{n+1}$, in this case, also approaches continually towards 1, as a limit, it is evident, that when n is an indefinitely large number, we shall have

$$\log. n = 2\text{M}(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \&c.).$$

Where the value of the right hand member, either with or without the factor 2M, must also be indefinite, or greater than any assignable number.

To this we may farther add, that every series, of which the terms decrease more rapidly than those of

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \&c.$$

must necessarily have some finite number for its sum, whether that sum can be determined or not.

For since the logarithm of n is finite, when n is not infinite, it is evident, that the series formed of the fractions

$$\frac{n-1}{n} + \frac{1}{2}(\frac{n-1}{n})^2 + \frac{1}{3}(\frac{n-1}{n})^3 + \frac{1}{4}(\frac{n-1}{n})^4 + \&c.$$

is susceptible of a limit, however near the fraction $\frac{n-1}{n}$ may otherwise approach to unity.

It may here also be observed, that the series

$$r^x = 1 + \frac{Ax}{1} + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \frac{A^4x^4}{1.2.3.4} + \&c.$$

before given, is always convergent, however great may be the value of x.

For since any two consecutive terms of this series can be represented by

$$\frac{k^n}{1.2.3 \ldots . n}, \text{ and } \frac{k^{n+1}}{1.2.3 \ldots . n(n+1)},$$

the ratio of which is $\frac{k}{n+1}$, it is plain, that by prolonging the series, we shall arrive at a term in which the denominator $n+1$ is greater than k.

Hence, the second of these terms being, in that case, less than the first, it is manifest, that all the ulterior terms will continually decrease.

Which considerations, it is obvious, are also equally applicable to any series, the numerators of the successive terms of which have a constant ratio, or that continually approach to such a ratio, whilst the denominators continually increase.

It may here also be farther remarked, that among the various logarithmic series hitherto given, no one has been found which proceeds according to the powers of the number, or that is of the form

$$\log. n = \text{A} + \text{B}n + \text{C}n^2 + \text{D}n^3 + \text{E}n^4 + \&c.$$

which circumstance, indeed, cannot take place; because, when $n=0$, the logarithm of n will become, as has been before shown, infinite, and negative, instead of its being $=\text{A}$ as it ought.

Neither can any series, of this kind, take the following form,

$$\log. n = \text{A} + \frac{\text{B}}{n} + \frac{\text{C}}{n^2} + \frac{\text{D}}{n^3} + \frac{\text{E}}{n^4} + \&c.$$

since, when n, in this case, is infinite, the series would be finite, being equal to A.

Another development, however, of the logarithm of n, may be found, which, from its remarkable

form, it will be here proper to explain: thus, taking as before,

$$\log.\ (1+n) = \text{M}\{n - \frac{n^2}{2} + \frac{n^3}{3} - \frac{n^4}{4} + \frac{n^5}{5} - \&c.\}$$

And

$$\log.\ (1+\frac{1}{n}) = \text{M}\{\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \frac{1}{5n^5} - \&c.\}$$

We shall have, on account of $\log.(1+\frac{1}{n}) = \log.\frac{1+n}{n} = \log.(1+n) - \log.n$, the following formula.

$$\log.(1+n) - \log.(1+\frac{1}{n}) = \log.n =$$

$$\text{M}\{(n-\frac{1}{n}) - \frac{1}{2}(n-\frac{1}{n^2}) + \frac{1}{3}(n-\frac{1}{n^3}) - \&c.\}$$

Or

$$\log.\ n = \text{M}\{(n-n^{-1}) - \frac{1}{2}(n-n^{-2}) + \frac{1}{3}(n-n^{-3}) - \frac{1}{4}(n-n^{-4}) - \&c.\}$$

Which series, though of no value in the actual computation of logarithms, is yet of great use, in leading to some interesting analogies between certain logarithmic functions, and those derived from the circle.

To what has been above shown, we may here, likewise, subjoin, for the satisfaction of the learner, the method employed by Lagrange, in his *Théorie des Fonctions*, for obtaining the first of the three principal logarithmic series, before found; which is different from that used in the former part of the present article.

For this purpose, let the primitive equation, before given, be taken under the form

$$a^x = y.$$

In which x is the logarithm of y for the base a.

Then, if $1+a-1$ be put in the place of a, and $\{(1+a-1)^n\}^{\frac{x}{n}}$ for its equal $(1+a-1)^x$, we shall have

$$y=\{(1+a-1)^n\}^{\frac{x}{n}}.$$

Where n is any arbitrary quantity, taken at pleasure, which will disappear of itself, in finding the value of y.

Hence, by expanding $(1+a-1)^n$, according to the manner of the binomial theorem, there will arise the series

$$1+n(a-1)+\frac{n(n-1)}{2}(a-1)^2+\frac{n(n-1)(n-2)}{2.3}(a-1)^3\&c.$$

Which, by arranging the terms according to the several powers of n, will give

$$(1+a-1)^n=1+\text{A}n+\text{B}n^2+\text{C}n^3+\text{D}n^4+\&c.$$

Where the coefficients A, B, C, &c. being all given in terms of a, it is easy to perceive, from the bare inspection of the first of these two series, that

$$\text{A}=(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\&c.$$

the quantity A being the same as the modulus before found, in the former part of this article.

Whence, making this substitution, in the former of these expressions, we shall have

$$y=(1+\text{A}n+\text{B}n^2+\text{C}n^3+\text{D}n^4\&c.)^{\frac{x}{n}}$$

Which being likewise developed, according to the manner of the binomial theorem, will give

$$y=1+\frac{x}{n}(\text{A}n+\text{B}n^2+\&c.)+\frac{x(x-n)}{2n^2}(\text{A}n+\text{B}n^2+\&c.)^2+$$
$$\frac{x(x-n)(x-2n)}{2.3n^3}(\text{A}n+\text{B}n^2+\&c.)^3+\&c.$$

Or, by effacing the powers of n, which are common to the numerators and denominators,

$$y = 1 + x(\text{A} + \text{B}n + \&\text{c.}) + \frac{x(x-n)}{2}(\text{A} + \text{B}n + \&\text{c.})^2 + \frac{x(x-n)(x-2n)}{2.3}(\text{A} + \text{B}n + \&\text{c.})^3 + \&\text{c.}$$

But as the quantity n, in this expansion, is arbitrary, it ought, from the nature of the function of y, to disappear in the expression of that function.

Whence, neglecting all the terms that are multiplied by the several powers of n, which, for the above reason, ought mutually to destroy each other, we shall have, simply,

$$y = a^x = 1 + \text{A}x + \frac{\text{A}^2}{2}x^2 + \frac{\text{A}^3}{2.3}x^3 + \frac{\text{A}^4}{2.3.4}x^4 + \&\text{c.}$$

Which is the same as the series before obtained in the first part of this article.

In like manner, if the original equation $a^x = y$, be now put under the identical forms

$$(1 + a - 1)^{nx} = (1 + y - 1)^n$$

where n is an arbitrary quantity, as before; we shall have, by developing each of the two members, according to the binomial theorem,

$$1 + nx(a-1) + \frac{nx(nx-1)}{2}(a-1)^2 + \frac{nx(nx-1)(nx-2)}{2.3}(a-1)^3 \&\text{c.} =$$

$$1 + n(y-1) + \frac{n(n-1)}{2}(y-1)^2 + \frac{n(n-1)(n-2)}{2.3}(y-1)^3 \&\text{c.}$$

Or, by suppressing unity, which is common to each of them, and dividing by n,

$$x(a-1)+$$
$$\frac{x(nx-1)}{2}(a-1)^2+\frac{x(nx-1)(nx-2)}{2.3}(a-1)^3+\&c.=$$
$$(y-1)+\frac{n-1}{2}(y-1)^2+\frac{(n-1)(n-2)}{2.3}(y-1)^3+\&c.$$

Where, since n has been taken at pleasure, it ought, as before observed, to disappear in each of the functions of x and y.

Whence, neglecting all the terms of these two expressions, that are multiplied by n, which, for the reason here assigned, ought to destroy each other, we shall have

$$(a-1)x-\frac{1}{2}(a-1)^2x+\frac{1}{3}(a-1)^3x-\frac{1}{4}(a-1)^4x\&c.=$$
$$(y-1)-\frac{1}{2}(y-1)^2+\frac{1}{3}(y-1)^3-\frac{1}{4}(y-1)^4\&c.$$

But, from what has been above shown,

$$A=(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\&c.$$

Therefore,

$$Ax=(y-1)-\frac{1}{2}(y-1)^2+\frac{1}{3}(y-1)^3-\frac{1}{4}(y-1)^4+\&c.$$

Or,

$$x=\log.y=\frac{(y-1)-\frac{1}{2}(y-1)^2+\frac{1}{3}(y-1)^3-\frac{1}{4}(y-1)^4\&c.}{A}$$

Which is the same as the logarithmic series found by the former method.

But as this formula is of no practical use, except when y differs but little from unity, the following method, which is also due to Lagrange, has been devised for rendering it convergent in all cases that may occur.

Thus, let there be taken the following similar expression,

$$\log. z =$$
$$\frac{1}{A}\{(z-1)-\frac{1}{2}(z-1)^2+\frac{1}{3}(z-1)^3-\frac{1}{4}(z-1)^4+\&c.\}$$

in which the modulus is represented by A, and y, in the former series, by z.

Then, if $\sqrt[r]{z}$, or $z^{\frac{1}{r}}$, which is equal $\frac{\log. z}{r}$, be substituted, in this last formula, for z, we shall have, by multiplying by r,

$$\log. z=\frac{r}{A}\{(z^{\frac{1}{r}}-1)-\frac{1}{2}(z^{\frac{1}{r}}-1)^2+\frac{1}{3}(z^{\frac{1}{r}}-1)^3-\&c.\}.$$

Where r may be any number, either positive or negative, taken at pleasure.

But whatever may be the numeral value of z, we can always find a root of it, of the degree r, such that $\sqrt[r]{z}$ shall be a number as little different from unity as we please.

Whence the preceding formula will give the value of log. z, in every case, to any degree of exactness that may be required.

If r be taken negatively, in which case $z^{\frac{1}{r}}=\frac{1}{z^{\frac{1}{r}}}$, the same series, by barely changing its signs, will become

$$\log. z=\frac{r}{A}\{(1-\frac{1}{z^{\frac{1}{r}}})+\frac{1}{2}(1-\frac{1}{z^{\frac{1}{r}}})^2+\frac{1}{3}(1-\frac{1}{z^{\frac{1}{r}}})^3+\&c.\}.$$

Where z being greater than unity, $z^{\frac{1}{r}}$ will also be greater than unity; and, consequently, we shall have

$$(z^{\frac{1}{r}}-1)=0, \text{ and } 1-\frac{1}{z^{\frac{1}{r}}}=0.$$

Also, taking a for the base of the system, the same formulæ will determine the value of the modulus A; since, in making log. $a=1$, we shall have

$$A=r\{(a^{\frac{1}{r}}-1)-\frac{1}{2}(a^{\frac{1}{r}}-1)^2+\frac{1}{3}(a^{\frac{1}{r}}-1)^3-\&c.\}.$$

Or

$$A=r\{(1-\frac{1}{a^{\frac{1}{r}}})+\frac{1}{2}(1-\frac{1}{a^{\frac{1}{r}}})^2+\frac{1}{3}(1-\frac{1}{a^{\frac{1}{r}}})+\&c.\}$$

It likewise readily appears, from what is above shown, that each of the two series, expressing log. z, must be convergent, when a root r of z is taken such, that $z^{\frac{1}{r}}-1$ is a fraction less than unity.

For, in that case, $1-\frac{1}{z^{\frac{1}{r}}}$ must be still less than $z^{\frac{1}{r}}-1$; since, by reduction it appears, that

$$1-\frac{1}{z^{\frac{1}{r}}}=\frac{z^{\frac{1}{r}}-1}{z^{\frac{1}{r}}}.$$

Wherefore, since the sums of the second and third, the fourth and fifth, &c. terms of the first of these series are negative, we shall have

$$\log. z=\frac{r}{A}(z^{\frac{1}{r}}-1).$$

And, on the contrary, because the terms of the second series are all positive, we shall have

$$\log. z=\frac{r}{A}(1-\frac{1}{z^{\frac{1}{r}}}).$$

Hence we have here two limits for the value of log. z; which, by increasing the magnitude of r, may be brought as near together as we please.

So that, when r is indefinitely great, either of the two expressions

$$\frac{r}{A}(z^{\frac{1}{r}}-1), \text{ or } \frac{r}{A}\left(1-\frac{1}{z^{\frac{1}{r}}}\right),$$

may, in that case, be considered as the true value of log. z.

In which last respect, it may be said, that a given number answers to an infinity of logarithms; since its infinite root must necessarily have an infinite number of different values.

Hence, suppose, by way of illustrating the use of the two series here given for finding log. z, that r is taken such, that the rth root of z shall contain unity only before the decimal point, and s ciphers after it; then, if we stop, in this extraction, at $2s$ decimals, it is plain that the term $(z^{\frac{1}{r}}-1)^2$, and, a fortiori, all the rest, will have no effect in the calculation; so that, in this case, we shall have

$$\log. z = \frac{r}{A}(z^{\frac{1}{r}}-1), \text{ and } A = r(a^{\frac{1}{r}}-1).$$

And if r, instead of being used in a general sense, be taken equal to some even power of 2, as, for instance, 2^{2n}, the operation may be effected solely by n successive extractions of the square root (*d*).

This method, however, will in general be found extremely laborious, on account of the great num-

(*d*) It was in this way that Briggs calculated most of the prime numbers in his table; in the performing of which he remarked, that if he stopped, in the successive extractions of the square root of any number, at twice as many decimals as there were ciphers following unity in the first part of the root, that the decimal part of this root was exactly half that of the preceding root; so that these decimal parts would have the same ratio to each other, as the logarithms of the roots themselves, as is evident from the above formulæ.

ber of extractions of the square root, which it will be requisite to make, in order to obtain a result true to a considerable number of decimals; but the series before given will serve to render it more simple and complete.

For, whatever may be the magnitude of z, it will be sufficient, in that case, to extract such a number of its square roots as will give $\sqrt[i]{z}$ a little greater than unity; as the different powers of $(\sqrt[i]{z}-1)$ will then be all fractions so much the less as those powers are greater; and, consequently, a certain number of terms of the series will give the logarithm as near the truth as may be required.

But a more commodious way than that of employing the successive extractions of the square root of z, in order to reduce $\sqrt[i]{z}-1$ to a fraction, is, when z is a large number, to find such a power a^m of some other number as is nearly equal to it, and put $z=a^m \pm d$; in which case

$$\log. z = \log. (a^m \pm d) =$$

$$\log. a^m(1 \pm \frac{d}{a^m}) = m \log. a + \log. (1 \pm \frac{d}{a^m}).$$

And, consequently, by using either of the series before given, we shall have

$$\log. z = m \log. a \pm \text{M}\{\frac{d}{a^m} \mp \frac{1}{2}(\frac{d}{a^m})^2 + \frac{1}{3}(\frac{d}{a^m})^3 \mp \&c.\}.$$

Or

$$\log. z =$$

$$m \log. a + 2\text{M}\{\frac{d}{2a^m \pm d} + \frac{1}{3}(\frac{d}{2a^m \pm d})^2 + \frac{1}{5}(\frac{d}{2a^m \pm d})^5 + \&c.\}.$$

Which series, in most cases of this kind, converges very rapidly.

To this we may farther add, that if, in the common exponential series before given,

$$n^x = 1 + \frac{A}{1}x + \frac{A^2}{1.2}x^2 + \frac{A^3}{1.2.3}x^3 + \frac{A^4}{1.2.3.4}x^4 + \&c.$$

$\frac{\log. n}{M}$ be substituted for its equal A, we shall have

$$n^x = 1 + \frac{\log. n}{M}x + \frac{1}{1.2}\left(\frac{\log. n}{M}\right)^2 x^2 + \frac{1}{1.2.3}\left(\frac{\log. n}{M}\right)^3 x^3 + \&c.$$

And, by taking x, in this expression, $= 1$, there will arise

$$n = 1 + \frac{\log. n}{M} + \frac{1}{1.2}\left(\frac{\log. n}{M}\right)^2 + \frac{1}{1.2.3}\left(\frac{\log. n}{M}\right)^3 + \&c.$$

Which last series gives the value of the number n, in terms of its logarithm, when the logarithm and the modulus of the system are known.

And if e be made to represent the number whose Neperian logarithm is 1; e being, in that case, the base of this system, we shall have

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2\,3.4} + \&c.$$

Also, since $\log. n = M\, l. n$, if x^m and r^m be successively substituted for n, in the last of the above expressions, we shall have

$$x^m = 1 + m\, l. x + \frac{1}{1.2}(m\, l. x)^2 + \frac{1}{1.2.3}(m\, l. x)^3 + \&c.$$

$$r^m = 1 + m\, l. r + \frac{1}{1.2}(m\, l. r)^2 + \frac{1}{1.2.3}(m\, l. r)^3 + \&c.$$

And, by subtracting the latter of these two series from the former, there will arise

$$x^m - r^m =$$

$$\frac{m(l. x - l. r)}{1} + \frac{m^2(l.^2 x - l.^2 r)}{1.2} + \frac{m^3(l.^3 x - l.^3 r)}{1.2.3} + \&c.$$

Which theorem was first given by Bossut, in his *Traité du Calcul Différential et Intégral*, in order

to explain an apparent paradox, which had been thought to attend one of the first principles of that doctrine.

OF FUNCTIONS.

HAVING treated pretty fully, in the preceding articles of the present work, on nearly every part of common algebra, which appeared the most deserving of attention, I shall here, by way of supplement, lay before the reader a slight sketch of the doctrine of functions; which is a new method of investigating the properties and relations of variable quantities, that has, of late, become one of the most fertile and important branches of analysis.

With this view, it will be proper to observe, that some of the earlier analysts comprehended under the denomination of the *functions* of a quantity, all the powers or roots of that quantity; and, afterwards, the same appellation was extended to the sums, products, quotients, &c. of those powers and roots.

At length, new ideas, arising out of the gradual progress of analysis, led to a more comprehensive definition of the term in question; which may be expressed as follows:

A function of one or more quantities, is an algebraical expression, into which those quantities enter, in any manner whatever, either mixed, or not, with known quantities.

Or, in other words, any quantity, of which the value depends upon one or several other quantities, is called a function of those quantities, whether we

know or not the operations that must be employed to pass from these to the former: thus

$$a^2 \pm x^2,\ (a^2 \pm bx^2)^{\frac{m}{n}}, a^x,\ a \log. x,\ \sin. x,\ \&c.$$

are all functions of the single quantity x; and

$$\frac{x}{a^x + xy},\ \frac{axy}{x^2 - y^2},\ \frac{xy}{(a + bx + cx^2)^{\frac{1}{2}}},\ \frac{a \sin. \frac{1}{2}x}{\cos. y},$$

are all functions of the two quantities x and y.

When the forms of these, or any other functions, are known, or given, they are commonly designated, for the sake of greater simplicity, by the symbols X, Y, or Z, or some other single letter.

But when they are indeterminate, or of an unknown form, the function, in that case, is represented by f, F, ϕ, or ψ: thus

$$fx,\ fx^2,\ f(a+bx),\ \&c.$$

denote the functions of x, x^2, $(a+bx)$, &c. respectively.

And if, in any investigation, other functions of the same quantities are to be indicated, they must be represented by a different symbol; as

$$\text{F}x,\ \text{F}x^2,\ \text{F}(a+bx),\ \&c.$$

taking care always to affix, in such instances, the same symbol to the same denomination of the function.

Also, if two functions of x, y, one in x and the other in y, be composed in the same manner, and have the same known quantities, they may be denoted, in any calculation, by the same symbol; thus

$$fx \text{ and } fy$$

are two like functions of x and y, which become identical in making $y = x$.

But if two functions, which are composed in the same manner, have different known quantities, they cannot, in general, be represented by the same characteristic, in the course of the same calculation.

Nevertheless, if any two functions of this kind differ only by having a known quantity a in one, and b in the other, they may be denoted by the same symbol, as follows:

$$f(x, a), \text{ and } f(y, b).$$

In which case a and b are supposed to enter into the expression of the function like unknown quantities; because, although they are constant, when taken separately, they must be regadred as if they were variable, when employed in this way.

In like manner, also, may the function of any two variable quantities, x, y, which are independent of each other, be designated under the form

$$f(x, y).$$

And so on for all other similar cases.

Functions are likewise distinguished by the denominations of algebraic and transcendental, according to the nature of the expression under which they are exhibited: thus,

Algebraic functions are such as result from the common operations of that science, or which can be expressed algebraically; as

$$a + bx, \ (a^2 - x^2)^{\frac{1}{2}}, \ a + (a^2 + x^2)^{\frac{1}{2}}, \ \&c.$$

And transcendental functions are such as cannot be expressed algebraically, by any limited number of terms; as

$$a^x, \ x^x, \ a \log. x, \ a \cos. x, \ \&c.$$

They are also divided into *explicit* and *implicit* functions, according to the nature of the equation by which they are expressed; thus in the equations

$$y = ax^2 + bx + c,$$
$$y = axz + bx^2 + cz^2,$$

y is an explicit function of x in the first, and of x and z in the second; because, when any particular values are given to the quantities x, z, the value of y becomes known.

But when the powers, or products, of the unknown quantities are mixed together, as in the two following equations,

$$x^3 + y^3 = axy,$$
$$x^3 + y^3 + z^3 = axz + byz + cxy,$$

x is said to be an implicit function of y in the first, and of y and z in the second; because the determination of x, when the latter are known, depends upon the solution of an algebraic equation.

The last of these denominations may also be applied to various other expressions, which do not appear under the form of an equation.

Thus, in the circle, the sine is an implicit function of the arc, although the state of analysis does not afford any means of expressing the relations of these two quantities.

It may here, also, be farther observed, that *symetrical* functions are such as do not change their values for any permutations that may be made in the quantities on which they depend.

Thus, if there be taken the two following equations,

$$x + y = p, \text{ and } xy = q,$$

from which, by substitution and transposition, we can readily obtain

$$x^2 - px + q = 0,$$
$$y^2 - py + q = 0,$$

it is obvious, that the two unknown quantities x and y are roots of the same equation; because they both enter, in the same manner, into the conditions of the question.

And if $2xy$, in the second of the proposed equations, be taken from the square of $x+y$ in the first, we shall have

$$x^2 + y^2 = p^2 - 2q.$$

Where it is plain, that x may be changed into y, or y into x, without altering the value of the expression.

In like manner, if α, β, γ, δ, ε, be made to denote the roots of an equation of the fifth power, the quantities

$$\alpha + \beta + \gamma + \delta + \varepsilon,$$
$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2,$$
$$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \varepsilon^3,$$
&c.

will be symetrical functions of the roots of that equation.

And the same may be shown of the sums of their products, when taken two by two, three by three, &c.; as in Art. (N) 3, p. 45.

To this we may farther add, that the functions of quantities may be frequently changed, either by the simple operations of algebra, or by substitution, into others of more convenient forms; thus, in the first case,

$$f(2-3x+x^2),\ f\left(\frac{2a^2}{a^2-x^2}\right),\ \text{and}\ f\left(\frac{k}{\sqrt{(1+x)^n}-x}\right),$$

may be transformed, by the first of these methods, into

$$f\{(1-x)(2-x)\},\ f\left(\frac{a}{a-x}+\frac{a}{a+x}\right),\ \text{and}$$
$$f\{\sqrt{(1+x^2)}+x\}.$$

Which latter expressions are respectively equivalent to the former.

The following easy examples, taken from a great variety of others that might be proposed, will also serve to illustrate the second of these methods.

1. Suppose it were required to render $\phi(a^2+x^2)^{\frac{1}{2}}$ rational.

This, by putting $x=\frac{a^2-y^2}{2y}$, will become $\phi\left(\frac{a^2+y^2}{2y}\right)$.

2. If it be required to render $\phi(a+bx)^{\frac{1}{2}}$ rational, we shall have, by substituting $bz^2-\frac{a}{b}$ for x, the equivalent expression $\phi(bz)$.

3. Again, if $\phi(a+bx)^{\frac{m}{n}}$ is to be rendered rational, this will become $\phi(z)^m$, by putting $(a+bx)^{\frac{m}{n}}=z^m$, which gives $x=\frac{z^n-a}{b}$.

4. And $\phi\left(\frac{a+bx}{c+dx}\right)^{\frac{m}{n}}$, may be transformed to $\phi(z)^m$, by putting $\left(\frac{a+bx}{c+dx}\right)^{\frac{m}{n}}=z^m$; which gives $x=\frac{a-cz^n}{dz^n-b}$.

5. In like manner, if x be any rational function of x, $\text{F}\left(\frac{\text{x}}{\sqrt{(1+x^2)}}\right)$ will become $\text{F}\left\{\frac{\text{x}(1-z^2)}{1+z^2}\right\}$, by putting $\sqrt{(1+x^2)}=1+zx$; which gives $x=\frac{2z}{1-z^2}$.

6. Also, supposing, as before, x to be any rational function of x, the following expression

$$F\left(\frac{x}{\sqrt{(a+bx+cx^2)}}\right),$$

may be made rational, by putting it under the form

$$F\left(\frac{x}{\delta\sqrt{(\alpha+\beta x+x^2)}}\right),$$

$$\text{where } \delta=\sqrt{c},\ \alpha=\frac{a}{c},\ \text{and } \beta=\frac{b}{c}.$$

For, making $\sqrt{(\alpha+\beta x+x^2)}=x+z$, we shall readily obtain,

$$x=\frac{\alpha-z^2}{2z-\beta},\ \text{and } \delta\sqrt{(\alpha+\beta x+x^2)}=\delta\left(\frac{\alpha-\beta z+z^2}{2z-\beta}\right).$$

Which, by substitution, will convert the proposed expression into a rational function in terms of z.

And, by following a similar method, various other transformations may be obtained, which are of great use in the Diophantine Analysis, and in finding the fluents of many fluxional expressions.

But the most valuable part of the doctrine of Functions, is that which has arisen out of the numerous discoveries and improvements that have been made in it by Lagrange, under whose hands it has assumed the form of a new science.

We can here, however, only enter into a few such details, respecting this branch of the subject, as may enable the reader to form some idea of the manner in which the first principles of the doctrine, as laid down by the author, in his two celebrated works, the *Théorie des Fonctions Analytiques*, and *Leçons sur le Calcul des Fonctions*, have been developed and applied.

For this purpose, let fx denote, agreeably to

what has been before shown, any function of the variable quantity x.

Then, if $x+i$ be substituted in the place of x, we shall have, according to all the known expansions of any binomial expression of this kind,

$$f(x+i)=fx+pi+qi^2+ri^3+\&c.$$

Where the coefficients p, q, r, &c. of the indeterminate quantity i, are new functions of x, which may be all derived from each other, according to a certain regular law.

For suppose the variable quantity x, in this expression, to be changed into $x+o$, o being any indeterminate quantity whatever, independent of i.

Then, since $f(x+i)$ becomes $f(x+i+0)$, it is evident that the result will be the same, whether we substitute $i+o$ for i, in the above series, or $x+o$ for x.

But, by the first of these substitutions, we shall have

$$fx+p(i+o)+q(i+o)^2+r(i+o)^3+\&c.$$

Or, taking only the two first terms of the several powers of $i+o$, which will be sufficient for the purpose intended,

$$\begin{aligned} fx+pi &+ qi^2 + ri^3 + si^4 + \&c. \\ &+ po + 2qio + 3ri^2o + 4si^3o + \&c. \end{aligned}$$

And, in order to make the second substitution, let $fx+po+\&c.$, $p+p'o+\&c.$, $q+q'o+\&c.$, be what fx, p, q, r, &c. become, in putting $x+o$ for x.

Then omitting, in the development, as before, all the terms that involve the square, and other higher powers of o, the same formula will become

$$fx + pi + qi^2 + ri^3 + si^4 + \&c.$$
$$+ po + p'io + q'i^2o + r'i^3o + \&c.$$

And, consequently, as the two results, thus obtained, will be identical, independently of i and o, which may be any quantities whatever, we shall have, by comparing the homologous terms, $p = p$, and

$$\begin{array}{l|c|l} 2q = p' & & q = \frac{1}{2}p' \\ 3r = q' & \text{Or} & r = \frac{1}{3}q' \\ 4s = r' & & s = \frac{1}{4}r' \\ \&c. & & \&c. \end{array}$$

Where p is the first function derived from fx, p' the first function derived from p, q' the first derived from q, r' the first derived from r; and so on.

Hence, if, for the sake of greater simplicity and uniformity, the first function derived from fx, be denoted by $f'x$, the first derived from $f'x$ by $f''x$, the first derived from $f''x$ by $f'''x$, &c. we shall have $p = f'x$; and, consequently,

$$\begin{array}{l|c|l} p' = f''x & & q = \frac{1}{2}f''x \\ q' = \frac{1}{2}f'''x & \text{Or} & r = \frac{1}{2.3}f'''x \\ r' = \frac{1}{2.3}f^{\text{iv}}x & & s = \frac{1}{2.3.4}f^{\text{iv}}x \\ \&c. & & \&c. \end{array}$$

The successive functions $f'x$, $f''x$, $f'''x$, &c. being here nothing more than the coefficients of i in the first terms of the developments of $f(x+i)$, $f'(x+i)$, $f''(x+i)$, &c.

Wherefore, if these values be substituted in the development of $f(x+i)$, above given, there will arise

$$f(x+i) = fx + if'x + \frac{i^2}{2}f''x + \frac{i^3}{2.3}f'''x + \frac{i^4}{2.3.4}f^{\text{iv}}x + \&c.$$

Which expression has the advantage of showing in what manner the terms of the series depend upon each other; and, above all, when we know the second term $if'x$, how all the rest of the terms may be obtained from it, by the same uniform process.

The first of these functions, or fx, is called the primitive function, with respect to $f'x$, $f''x$, &c, which are derived from it; and the latter are called derived functions, with respect to the first.

Thus $f'x$ is called the first derived function, or simply the first function; the second $f''x$, which is derived from the last, the second derived function, or simply ths second function; and so on.

In like manner, if y be any function whatever of x, and y', y'', y''', &c. be the first, second, third, &c. derived functions, the value of y, when x becomes $x+i$, will be

$$y+iy'+\frac{i^2}{2}y''+\frac{i^3}{2.3}y'''+\text{ \&c.}$$

So that, although the several orders of derived functions owe their origin to the development of the primitive function, when augmented by some indeterminate quantity i, they are wholly independent of that quantity, which is of use chiefly in ascertaining their formation.

Hence, when we know, in any case, how to pass from the primitive function to the first derived function, we can abstract from the development, and consider the derivations of these functions as a new species of algebraical operation, more general and extensive than that of the raising of powers, or extraction of roots.

Thus, let there be taken, as one of the easiest examples of this method, $fx = x^m$, which gives

$$f(x+i) = (x+i)^m.$$

Then, since it can be shown, from the simple principles of algebra, that the two first terms of the mth power of $x+i$ are

$$x^m + mx^{m-1}i,$$

whatever may be the value of m, it follows, from what has been above shown, that

$$f'x = mx^{m-1},\ f''x = m(m-1)x^{m-2},$$
$$f'''x = m(m-1)(m-2)x^{m-3},\ \&c.$$

And, therefore, by substituting these quantities in the general series

$$f(x+i) = fx + if'x + \frac{i^2}{2}f''x + \frac{i^3}{2 \cdot 3}f'''x + \&c.$$

before found, we shall have, for the case $fx = x^m$, the series

$$f(x+i) =$$
$$x^m + mx^{m-1}i + \frac{m(m-1)}{2}x^{m-2}i^2 + \frac{m(m-1)(m-2)}{2 \cdot 3}x^{m-3}i^3 + \&c.$$

which is the known formula of the binomial, or the development of $(x+i)^m$, for any value of m, whether integral or fractional, positive or negative.

Again, let there be taken, as a second example of this kind, the exponential expression $fx = a^x$, which gives

$$f(x+i) = a^{x+i} = a^x a^i.$$

Then, if there be put, in this case, $a = 1 + b$, we shall have, by the formula before demonstrated,

$$a^i = (1+b)^i = 1 + ib + \frac{i(i-1)}{2}b^2 + \frac{i(i-1)(i-2)}{2 \cdot 3}b^3 + \&c.$$

And, consequently, by ordering the terms ac-

cording to the powers of i, the two first terms of the series will be

$$1+(b-\tfrac{1}{2}b^2+\tfrac{1}{3}b^3-\tfrac{1}{4}b^4+\&c.)i.$$

Or taking, for the sake of greater simplicity,

$$\left.\begin{array}{c} b-\frac{1}{2}b^2+\frac{1}{3}b^3-\frac{1}{4}b^4+\&c. \\ \text{or} \\ (a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\&c. \end{array}\right| = \text{A},$$

the same two terms may be denoted by $1+\text{A}i$; A being what is usually considered as the modulus of the exponential of which a is the base.

Hence, multiplying this last expression by a^x, we shall have

$$a^x+\text{A}a^xi,$$

for the two first terms of the development of a^{x+i}; where $\text{A}a^x$ is the first function derived from a^x.

And, therefore, by repeating the same operation, the second function, which is derived from this, will be $\text{A}\times\text{A}a^x$, or A^2a^x, the third, or that derived from the last, A^3a^x, &c.; so that making $fx=a^x$, we shall have

$$f'x=\text{A}a^x,\ f''x=\text{A}^2a^x,\ f'''x=\text{A}^3a^x,\ \&c.$$

Whence, if these values be substituted in the development of $f(x+i)$, the formula before given will become

$$a^{x+i}=a^x+\text{A}a^xi+\frac{\text{A}^2}{2}a^xi^2+\frac{\text{A}^3}{2.3}a^xi^3+\&c.$$

And, consequently, by dividing each side of this equation, by a^x, we shall have

$$a^i=1+\text{A}i+\frac{\text{A}^2}{2}i^2+\frac{\text{A}^3}{2.3}i^3+\frac{\text{A}^4}{2.3.4}i^4+\&c.$$

Which last series serves to show how any power of a quantity may be converted into a series, that

shall follow the order of the successive powers of its exponent.

The same series also enables us to determine the value of the base a in terms of the modulus A; for, by taking $i=1$, we shall have

$$a=1+\mathrm{A}+\frac{1}{2}\mathrm{A}^2+\frac{1}{2.3}\mathrm{A}^3+\frac{1}{2.3.4}\mathrm{A}^4+\&c.$$

And when A, in this last equation, is $=1$, the value of a will be expressed by the simple series

$$1+1+\frac{1}{2}+\frac{1}{2.3}+\frac{1}{2.3.4}+\&c.$$

the approximate sum of which, according to what has been shown in the preceding article, will be $=$

$$2.718281828459.$$

Which number is commonly represented by e, being the base of the exponential whose modulus is unity.

Hence, taking, agreeably to this mode of notation, $fx=e^x$, we shall likewise have $f'x=e^x$, $f''x=e^x$, &c.; and, consequently,

$$e^i=1+i+\frac{1}{2}i^2+\frac{1}{2.3}i^3+\frac{1}{2.3.4}i^4+\&c.$$

Also, if in the equation $a^i=1+\mathrm{A}i+\frac{1}{2}\mathrm{A}^2i^2+\&c.$ above found, i be taken $=\frac{1}{\mathrm{A}}$, we shall have

$$a^{\frac{1}{\mathrm{A}}}=1+1+\frac{1}{2}+\frac{1}{2.3}+\frac{1}{2.3.4}+\&c.=e.$$

From which equation it appears, that the relations of the three constant quantities a, A, and e, may be expressed by

$$a^{\frac{1}{\mathrm{A}}}=e,\ a=e^{\mathrm{A}},\ \text{or}\ \frac{1}{a}=e^{-\mathrm{A}}.$$

Where it is plain, that, taking A negatively, a will be changed into $\frac{1}{a}$; and, consequently, the former equation $A=(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-$ &c. will become

$$A=\frac{a-1}{a}+\frac{1}{2}\left(\frac{a-1}{a}\right)^2+\frac{1}{3}\left(\frac{a-1}{a}\right)^3+\&c.$$

Which latter series will give the value of A more readily than the former, when a is a number greater than unity.

Thus, for instance, if a, in this case, be taken $=10$, we shall have

$$A=\frac{9}{10}+\frac{1}{2}\left(\frac{9}{10}\right)^2+\frac{1}{3}\left(\frac{9}{10}\right)^3+\frac{1}{4}\left(\frac{9}{10}\right)^4+\&c.$$

Which, by calculating a sufficient number of the terms, will give

$$A=2.302585092994.$$

Lastly, let there be taken $x=a^{\log.x}$, or $fx=\log.x$; which gives

$$x=a^{fx}.$$

Then, putting $x+i$ in the place of x, we shall have $f(x+i)=\log.(x+i)$, or

$$x+i=a^{f(x+i)}.$$

The two members of which equation ought to be identical, whatever may be the value of i.

But, by the general development before given, we have

$$f(x+i)=fx+if'x+\frac{i^2}{2}f''x+\frac{i^3}{2.3}f'''x+\&c.$$

Or, putting $if'x+\frac{i^2}{2}f''x+\frac{i^3}{2.3}f'''x+\&c.=z$, the same formula will become

$$f(x+i)=fx+z.$$

Or

$$x+i=a^{fx+z}=a^{fx}\times a^{z}=xa^{z}.$$

And, consequently, by dividing $x+i$ and xa^z by x, we shall have

$$1+\frac{i}{x}=a^{z}.$$

But according to the formula above given, we have, in general,

$$a^{z}=1+\mathrm{A}z+\frac{\mathrm{A}^2}{2}z^2+\frac{\mathrm{A}^3}{2.3}z^3+\&c.$$

Or

$$1+\frac{i}{x}=1+\mathrm{A}z+\frac{\mathrm{A}^2}{2}z^2+\frac{\mathrm{A}^3}{2.3}z^3+\&c.$$

Whence, putting for z its former value, and ordering the terms according to the powers of i, there will arise the identical equation

$$\frac{i}{x}=i\mathrm{A}f'x+\frac{i^2}{2}\{\mathrm{A}^2(f'x)^2+\mathrm{A}f''x\}+\&c.$$

From which, by a comparison of the terms, we shall have

$$\frac{1}{x}=\mathrm{A}f'x,\ \text{or}\ f'x=\frac{1}{\mathrm{A}x}.$$

And, consequently, by taking the other derived functions, according to the general rule,

$$f''x=-\frac{1}{\mathrm{A}x^2},\ f'''x=\frac{2}{\mathrm{A}x^3},\ \&c.$$

Whence, substituting these in the development of $f(x+i)$, we shall have

$$\log.\,(x+i)=\log.\,x+\frac{i}{\mathrm{A}x}-\frac{i^2}{2\mathrm{A}x^2}+\frac{i^3}{3\mathrm{A}x^3}-\&c.$$

But, by the theory of logarithms, laid down in the last article.

$$\log.\,(x+i)-\log.\,x=\log.\,\left(\frac{x+i}{x}\right)=\log.\,\left(1+\frac{i}{x}\right).$$

Whence, if $\frac{i}{x}$, in the preceding formula, be put $=y$, we shall have

$$\log.(1+y)=\frac{1}{A}(y-\frac{1}{2}y^2+\frac{1}{3}y^3-\frac{1}{4}y^4+\&c.).$$

Or making $1+y=z$, which gives $y=z-1$, this last equation will become

$$\log.z=$$
$$\frac{1}{A}\{(z-1)-\frac{1}{2}(z-1)^2+\frac{1}{3}(z-1)^3-\frac{1}{4}(z-1)^4+\&c.\}$$

Which is the same as the series before found, for log. z, in the article before quoted.

Many other applications of this doctrine, of a much more extensive and interesting nature, may be found in the two learned works mentioned in the former part of this article; but, as they relate chiefly to some of the higher branches of analysis, they could not, with propriety, be introduced into a performance like the present, which is meant to be purely algebraical (*e*).

(*e*) The principal object of the author, in the works here referred to, is to establish the Differential Calculus, or Method of Fluxions, upon pure analytical principles, independently of any assumption of infinitely small quantities, or such as are supposed to be generated by motion; which he justly regards as involving notions and difficulties wholly foreign to the nature of the subject. But although the new point of view, under which these important doctrines are here exhibited, possesses the great advantage of being free from the obscurities and metaphysical subtleties that have hitherto attended them, it must yet be confessed, that the mode he has followed, in treating of their first principles, is neither so simple nor unobjectionable as could be wished; particularly in the expansion of his fundamental formula, $f(x+i)$, which he has generalised too hastily; as we can, evidently, form no idea how such a development is to be effected, without a reference to the particular cases in which that function has been expanded.

OF ELIMINATION.

ELIMINATION is the method of exterminating all the unknown quantities, except one, from two, three, or more, given equations, in order to reduce them to a single, or final equation, which shall contain only the remaining unknown quantity and certain known coefficients (*f*).

This may be done, when the proposed equations are all simple ones, by the rules which have been laid down for that purpose, in Art. (o), Vol. I, p. 96; but if they are of the second, third, or other higher orders, the elimination is frequently attended with great difficulties.

In these cases, when only two equations of this kind are given, the most natural method appears to be that of finding the value of the unknown quantity, which is to be eliminated, in one of them, and then substituting that value in the other; by which means we shall obtain a single equation, containing only one unknown quantity.

Thus, if it were required to eliminate x in the two equations

$$x^2+y^2-5=0,$$
$$x^2+\frac{5}{2}xy+y^2=0,$$

(*f*) The subject here treated of is, properly, a branch of the general doctrine of equations, laid down in Art. (N), Vol. II, p. 38, which should have been introduced under that head; but being of a very intricate nature, it was judged best to give such an account of it, as the limits of the work would permit, in a separate article.

we shall have, from the first, $x=\sqrt{(5-y^2)}$; which value, being substituted in the second, gives

$$y\sqrt{(5-y^2)}=-2, \text{ or}$$
$$y^2(5-y^2)=4.$$

And, therefore, by multiplication and transposition, there will arise

$$y^4-5y^2+4=0;$$

which is the resulting or final equation required.

Also, if it were required to eliminate x, in the two equations

$$x^2-2yx+8=0,$$
$$x^2-2y^2+14=0,$$

we shall have, from the second, $x=\sqrt{(2y^2-14)}$; which value, being substituted in the first, gives

$$2y^2-14-2y\sqrt{(2y^2-14)}+8=0, \text{ or}$$
$$2y^2-6=2y\sqrt{(2y^2-14)}.$$

And, consequently, by involution and transposition, there will arise

$$y^4-8y^2-9=0;$$

which is the resulting or final equation in this case.

And if y and x be determined, in either of these examples, according to the common rules, their corresponding values will be found to answer the conditions of the question.

But this method, it is evident, cannot always be used, on account of our having no general rule for determining the roots of equations above those of the fourth degree.

And besides, if the proposed equations were only of the third degree, instead of the second, the calculation, in this way, would be found extremely

embarrassing, from its being necessary to free the final equation from surds.

Some of the earlier analysts, therefore, had recourse, upon these occasions, to a different mode of proceeding; which may be illustrated as follows:

Let it be required, for instance, to eliminate x, from the two following equations of the second degree,

$$x^2 + \text{P}x + \text{Q} = 0,$$
$$x^2 + \text{P}'x + \text{Q}' = 0;$$

where P, Q, P′, Q′, are supposed to be certain given functions of y.

Then, by subtracting the second equation from the first, we shall have

$$(\text{P} - \text{P}')x + \text{Q} - \text{Q}' = 0, \text{ or}$$
$$x = -\frac{\text{Q} - \text{Q}'}{\text{P} - \text{P}'}.$$

And, consequently, if this value be substituted in either of the two proposed equations, as, for example, in the first, there will arise

$$\frac{(\text{Q} - \text{Q}')^2}{(\text{P} - \text{P}')^2} - \frac{\text{P}(\text{Q} - \text{Q}')}{\text{P} - \text{P}'} + \text{Q} = 0.$$

Or, by clearing the expression of its denominators, and simplifying the result

$$(\text{Q} - \text{Q}')^2 + (\text{P} - \text{P}')(\text{PQ}' - \text{QP}') = 0;$$

which last formula, it is obvious, will give the final equation in y, by barely substituting for P, P′, Q, Q′, their particular values.

Also, if it were required to eliminate x, in the two following equations of the third degree,

$$\text{P}x^3 + \text{Q}x^2 + \text{R}x + \text{S} = 0,$$
$$\text{P}'x^3 + \text{Q}'x^2 + \text{R}'x + \text{S}' = 0,$$

we shall have, by multiplying the first by P′, and the second by P, and subtracting the latter result from the former,

$$(P'Q - PQ')x^2 + (P'R - PR')x + P'S - PS' = 0. \ldots (1)$$

And, by again multiplying the first of the same two equations by S′ and the second by S, and subtracting as before, there will arise

$$(S'P - SP')x^3 + (S'Q - SQ')x^2 + (S'R - SR')x = 0.$$

Or

$$(S'P - SP')x^2 + (S'Q - SQ')x + (S'R - SR') = 0. \ldots (2)$$

Whence, if equations (1) (2), which are now each of the second degree with respect to x, be represented by

$$p x^2 + q x + r = 0,$$
$$p' x^2 + q' x + r' = 0,$$

we can obtain, by treating them in the same manner as the former, two other equations of the first degree with respect to x; from which the final equation in y may be readily determined.

But this method, also, though far less laborious than the first, is still very incomplete, as it generally gives the final equation of a higher degree than it ought, and throws no light upon the manner of finding the value of the unknown quantity that is eliminated.

In order, therefore, to remedy these defects, it became necessary to devise some other mode of investigation, that should be less complex and uncertain in its operations, and more directly applicable to the purpose intended.

This was done by considering, that any two

equations of the kind here mentioned, which are compatible with each other, must have one, or more, of the values of x and y common to them both, or otherwise they could not subsist together at the same time; as is evident from the examples above given.

Hence, since the left hand members of any two equations of this kind, by Art. (N), p. 38, must have a common divisor, if this be determined according to the method made use of for that purpose, in Art. (F), Vol. I, p. 28, by continuing the operation till we arrive at a remainder independent of one of the unknown quantities, as x, this remainder being put $=0$, will give the final equation in terms of the other quantity y.

Thus, if there be given the two following equations,

$$x^3+3x^2y+3xy^2-98=0,$$
$$x^2+4xy-2y^2-10=0,$$

their greatest common divisor, found by following the process above mentioned, will be

$$(9y^2+10)x-2y^3-10y-98.$$

And the last remainder, which is independent of x, when put $=0$, will be

$$43y^6+345y^4-1960y^3+750y^2-2940y-4302=0.$$

From which equation it will be readily found, by trial, that

$$y=3,$$

Whence, putting the common divisor, above found, or the remainder preceding the last, also $=0$, and substituting 3 for y, we shall have, in this case,

$$(81+10)x-54-30-98=0, \text{ or}$$
$$91x-182=0; \text{ where } x=2.$$

But if, in any question of this kind, the value of y, obtained from the final equation, should, of itself, render the above mentioned remainder $=0$, we must then have recourse to the next preceding remainder; which, in that case, will be the common divisor; and, being a quadratic, will give, when put $=0$, two values of x, corresponding to that of y.

And if this remainder, likewise, should be annihilated, by the same substitution for y, we must then take the remainder preceding that; which will now be the common divisor; and, being a cubic, will give, by putting it $=0$, three values of x, answering to that of y; and so on.

Which mode of reasoning, it is evident, will also apply to any two equations whatever, of this kind,

$$x^m+\text{P}x^{m-1}+\text{Q}x^{m-2}+\ldots\ldots+\text{T}x+\text{V}=0,$$
$$x^n+\text{P}'x^{n-1}+\text{Q}'x^{n-2}+\ldots\ldots+\text{T}'x+\text{V}'=0.$$

Where the second unknown quantity y is enveloped in the coefficients P, Q, &c. and P′, Q′, &c. in such a manner, that the latter, when the equations are complete, are of the form

$$\text{P}=a+by,\quad \text{Q}=c+dy+ey^2,$$
$$\text{R}=f+gy+hy^2+ky^3;$$
$$\text{P}'=a'+b'y,\quad \text{Q}=c'+d'y+e'y^2,$$
$$\text{R}=f'+g'y+h'y^2+k'y^3.$$

In which case it is only necessary, as in the former example, to find the greatest common divisor of their two left hand members, and then

putting the remainder, which is independent of x, $=0$, for the final equation in y; and afterwards ascending to the preceding remainders, for the common divisor that ought to give the value of x.

It may here be observed, however, that Euler, instead of finding the common divisor in this way, employed a more commodious method of proceeding; which may be shown thus:

Let the two proposed equations, that are to be resolved, be

$$x^2 + \text{P}x + \text{Q} = 0,$$
$$x^3 + \text{P}'x^2 + \text{Q}'x + \text{R}' = 0,$$

where P, Q, P′, Q′, R′, are certain known functions of y, as before.

Then, if $x - \alpha$ be made to denote the factor which ought to be common to the first two members of these equations, we shall have, by the following assumption of the remaining factors,

$$x^2 + \text{P}x + \text{Q} = (x - \alpha)(x + s),$$
$$x^3 + \text{P}'x^2 + \text{Q}'x + \text{R}' = (x - \alpha)(x^2 + px + q);$$

where s, p, q, are unknown functions of the coefficients, involving the several powers of y, that are to be determined.

And, consequently, by eliminating $x - \alpha$, in each of these last equations, we shall have

$$\frac{x^2 + \text{P}x + \text{Q}}{x + s} = \frac{x^3 + \text{P}'x^2 + \text{Q}'x + \text{R}}{x^2 + px + q}.$$

Or

$$(x^2 + \text{P}x + \text{Q})(x^2 + px + q) = (x^3 + \text{P}'x^2 + \text{Q}'x + \text{R}')(x + s).$$

Which last expression, by actual multiplication and simplifying the result, will become

$$x^4 + (\text{P} + p)x^3 + (\text{Q} + p\text{P} + q)x^2 + (p\text{Q} + q\text{P})x + q\text{Q} =$$
$$x^4 + (\text{P}' + s)x^3 + (\text{Q}' + s\text{P}')x^2 + (\text{R}' + s\text{Q}')x + s\text{R}',$$

Hence, by comparing the coefficients of the homologous terms, or like powers of x, there will arise

$$\mathrm{P}+p=\mathrm{P}'+s,$$
$$\mathrm{Q}+p\mathrm{P}+q=\mathrm{Q}'+s\mathrm{P}',$$
$$p\mathrm{Q}+q\mathrm{P}=\mathrm{R}'+s\mathrm{Q}',$$
$$q\mathrm{Q}=s\mathrm{R}'.$$

And as we have here four equations, and only three unknown quantities, p, q, s, we can exterminate those which rise to the first degree; and by that means obtain an equation containing only the quantities P, Q; P', Q', R', which, therefore, will be the final equation in y.

But this theory, as far as regards the solution of two equations containing two unknown quantities, may be rendered more simple by means of the symetrical functions of their roots, treated of in Art. (N) 4, p. 51.

Thus, let there be taken, as before, the two complete equations (g)

$$x^m+\mathrm{P}x^{m-1}+\mathrm{Q}x^{m-2}+\ \ldots\ldots+\mathrm{T}x+\mathrm{V}=0,\ldots(1)$$
$$x^n+\mathrm{P}'x^{n-1}+\mathrm{Q}'x^{n-2}+\ \ldots\ldots+\mathrm{T}'x+\mathrm{V}'=0\ldots(2)$$

And suppose equation (1) to be resolved, and that its roots are $x=\alpha$, $x=\beta$, $x=\gamma$, &c.; α, β, γ,

(g) Any equation of this kind, is said to be complete, when it contains all the powers and products of x and y, that do not surpass the degree by which the equation is denoted; reckoning the powers of the products as formed by the sum of their exponents: thus

$$x^m+(a+by)x^{m-1}+(c+dy+ey^2)x^{m-2}+\ldots(p+qy+\ldots ty^m)=0$$

is a complete equation of the mth degree, as will be found by multiplying the factors, of which its several terms are composed, together.

&c. being functions of the other unknown quantity y.

Then, since these roots must also be roots of equation (2), we shall have, by the separate substitution of them in that equation, the following results:

$$\alpha^n + P'\alpha^{n-1} + Q'\alpha^{n-2} + \ldots\ldots + T'\alpha + V' = 0 \ldots (3)$$
$$\beta^n + P'\beta^{n-1} + Q'\beta^{n-2} + \ldots\ldots + T'\beta + V' = 0 \ldots (4)$$
$$\gamma^n + P'\gamma^{n-1} + Q'\gamma^{n-2} + \ldots\ldots + T'\gamma + V' = 0 \ldots (5)$$
$$\&c. \qquad\qquad \&c.$$

Which are all different equations in y, whose roots, when combined with those of $x=\alpha$, $x=\beta$, $x=\gamma$, &c. ought to satisfy each of the former equations (1) (2), by reducing them to 0.

And as the several values of y, deduced from these equations, must be roots of the final equation, this may be obtained by multiplying them together, and putting their product $=0$; since the result, in that case, will evidently involve all those roots.

But as this product will not be changed, for any permutations that may be made in the order of the quantities α, β, γ, &c. which all concur, in the same manner, in its formation, it follows, that these may be all obtained in terms of the coefficients P, Q, R, &c. of equation (1), according to the method laid down for that purpose in the article before referred to.

Thus, as an exemplification of this mode of proceeding, let there be given the two equations

$$x^2 + axy + by^2 + cx + dy + e = 0,$$
$$x^2 + a'xy + b'y^2 + c'x + d'y + e' = 0,$$

from which it is required to find the final equation in y.

Then, if there be put, in this case, $p = ay + c$, $q = by^2 + dy + e$, $p' = a'y + c'$, and $q' = b'y^2 + d'y + e'$, the proposed equations will become of the forms

$$x^2 + px + q = 0,$$
$$x^2 + p'x + q' = 0.$$

Wherefore, supposing α and β to be the two values, or roots of x, deduced from the first of these equations, we shall have, by substituting them in the second,

$$\alpha^2 + p'\alpha + q' = 0,$$
$$\beta^2 + p'\beta + q' = 0.$$

And, consequently, as these last equations ought to concur equally in the formation of the final equation, they must be so combined that the latter may contain them both.

Hence, multiplying them together, in order to obtain this result, we shall have, for their product,

$$\alpha^2\beta^2 + p'(\alpha\beta^2 + \alpha^2\beta) + p'^2\alpha\beta + q'(\alpha^2 + \beta^2) + p'q'(\alpha + \beta) + q'^2 = 0.$$

But from what has been shown, in Art. (N) 3, p. 44, respecting the relations of the roots of equations, it appears, that

$$\alpha^2\beta^2 = q^2,\ \alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha + \beta) = -pq;$$
$$(\alpha + \beta) = -p,\ \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = p^2 - 2q.$$

Whence, by substituting these values for their equals, in the above equation, we shall have

$$q^2 - p'pq + p'^2q + p^2q' - 2qq' - pp'q' + q'^2 = 0.$$

Or, since $q^2 - 2qq' + q'^2 = (q - q')^2$, and $-p'pq + p'^2q + p^2q' - pp'q' = (pq' - p'q)(p - p')$, the same equation will become of the form

$$(q - q')^2 + (pq' - p'q)(p - p') = 0;$$

which, therefore, by putting the particular values

of y for p, q; p', q', will be the final equation required (*h*).

If three equations of this kind be given, containing three unknown quantities, as

$$(1) \ldots F(x, y, z) = 0, \quad (2) \ldots f(x, y, z) = 0,$$
$$(3) \ldots \phi(x, y, z) = 0,$$

they may be resolved by first eliminating z in equations (1), (2), and then in equations (1), (3), when there will remain only two equations in x and y, which may be treated as above.

Thus, for example, let the three proposed equations be

$$x + y + z - 6 = 0, \quad z^2 - x^2 - y^2 - 4 = 0, \text{ and}$$
$$z^2 + xy - y^2 - 7 = 0.$$

Then, by eliminating z in the first and second of these equations, and in the first and third, we shall have

$$(x - 6)y - 6x + 16 = 0, \text{ and}$$
$$(3x - 12)y + (x^2 - 12x + 29) = 0.$$

And, by again eliminating y in these two last equations, there will arise

$$x^3 - 19x + 18 = 0,$$

for the final equation in x; in which x is evidently $= 1$.

And, if this value be substituted for x, in either of

(*h*) It might have been here farther shown, that the final equation, resulting from the elimination of one of the unknown quantities in any two equations of this kind, of the degrees m and n, can never rise higher, when properly resolved, than the mnth degree; but the demonstration is too complicated and abstruse to be given in a work like the present.

the two equations in x and y, we shall have $y=2$; and 1 and 2 being put for x and y, in the three first equations, will give $z=3$: which values of x, y, and z will be found to satisfy each of the proposed equations.

But it must be observed, that by this successive elimination, the three given equations do not concur in the same manner, in the formation of the final equation.

Equation (1), for instance, is employed twice, while equations (2) and (3) are only employed once; by which it happens, that the result obtained, in this way, frequently involves a factor that does not belong to the question.

To what is here said, we may farther add, that Cramer, at the end of his *Introduction à l'Analyse des Lignes Courbes*, 1750, p. 656, *et seq.*, has given a method of eliminating all the unknown quantities, except one, from any number of simple equations, which is much more commodious in practice than that laid down in Art. (o), p. 79; which method, as improved by Bezout in his *Théorie générale des Equations Algébriques*, 1779, may be elucidated as follows (*i*).

Let there be proposed, for instance, as one of

(*i*) The rule here given appears to have been obtained by induction, from observing the law of the formation of the numerators and denominators of the fractions expressing the values of the unknown quantities, in such cases of this kind as had been previously resolved by other methods; but the principles upon which it is founded are not sufficiently evident to lead to a general demonstration of the theorem.

the easiest cases of this kind, the two simple equations

$$ax + by + c = 0,$$
$$a'x + b'y + c' = 0.$$

Then, if the absolute terms c, c' of these equations be each multiplied by some unknown quantity t, we shall have

$$ax + by + ct = 0,$$
$$a'x + b'y + c't = 0.$$

And if x, in the product of the three unknown quantities xyt, be changed into a, y into b, and t into c, there will arise, by changing the signs of the even terms,

$$ayt - bxt + cxy.$$

Also, if x, in this last expression, be again changed into a', y into b', and t into c', the result, by changing the second, fourth, &c. terms, as before, will be

$$ab't - ac'y - a'bt + bc'x + a'cy - b'cx.$$

Or, by collecting the coefficients of the unknown quantities t, y, x,

$$(a'b - ab')t - (ac' - a'c)y + (bc' - b'c)x.$$

From which expression, according to the principles he has laid down, in the article above referred to, we shall have

$$x = \frac{bc' - b'c}{ab' - a'b}, \text{ and } y = -\frac{ac' - a'c}{ab' - a'b}.$$

The values of the unknown quantities, in each of the proposed equations, being always the fractions, whose numerators are their coefficients, in the last of these expressions thus obtained, and their denominators the coefficient of t.

Again, let there be taken, as another example, the three simple equations,

$$ax + by + cz + d = 0,$$
$$a'x + b'y + c'z + d' = 0,$$
$$a''x + b''y + c''z + d'' = 0.$$

Then, multiplying the absolute terms d, d', d'', by t, as before, we shall have

$$ax + by + cz + dt = 0,$$
$$a'x + b'y + c'z + d't = 0,$$
$$a''x + b''y + c''z + d''t = 0.$$

And, consequently, by changing x in the product, $xyzt$ into a, y into b, z into c, and t into d, there will arise, for the first line, by changing the signs of the even terms,

$$ayzt - bxzt + cxyt - dxyz.$$

And by again changing x into a', y into b', z into c', and t into d', the second line, by observing the same rule for the signs, will be

$$ab'zt - ac'yt + ad'yz - a'bzt + bc'xt - bd'xz + ca'yt - cb'xt + cd'xy - a'dyz + db'xz - dc'xy.$$

Or, by collecting the coefficients of zt, yt, yz, &c. properly together,

$$(ab' - a'b)zt - (ac' - a'c)yt + (ad' - a'd)yz + (bc' - b'c)xt - (bd' - b'd)xz + (cd' - c'd)xy.$$

Also, by again changing x into a'', y into b'', z into c'', and t into d'', and observing the same rule for the signs as before, the third line will be

$$+\{(ab' - a'b)c'' - (ac' - a'c)b'' + (bc' - bc')a''\}t$$
$$-\{(ab' - a'b)d'' - (ad' - a'd)b'' + (bd' - b'd)a''\}z$$
$$+\{(ac' - a'c)d'' - (ad' - a'd)c'' + (cd' - c'd)a''\}y$$
$$-\{(bc' - b'c)d'' - (bd' - b'd)c'' + (cd' - c'd)b''\}x$$

Whence, according to the principle before mentoned, we shall have

$$x = \frac{-(bc'-b'c)d''-(bd'-b'd)c''+(cd'-c'd)b''}{(ab'-a'b)c''-(ac'-a'c)b''+(bc'-b'c)a''}$$

$$y = \frac{+(ac'-a'c)d''-(ad'-a'd)c''+(cd'-c'd)a''}{(ab'-a'b)c''-(ac'-a'c)b''+(bc'-b'c)a''}$$

$$z = \frac{-(ab'-a'b)d''-(ad'-a'd)b''+(bc'-b'c)a''}{(ab'-a'b)c''-(ac'-a'c)b''+(bc'-b'c)a''}$$

And in the same way may this process be employed for four, five, or more equations; observing when any of the signs of the coefficients of the unknown quantities are $-$, to employ a contrary sign to that which the rule prescribes; and if any term in the proposed equations be wanting, its place may be supplied by a fictious coefficient, which can afterwards be rejected in the result.

What has been said, in this and the former part of the present article, will enable the learner to resolve any number of simple equations, of the kind above mentioned, as well as any two equations of any degree whatever; but if these methods be applied to three or more equations of the higher orders, they will be found subject to numerous inconveniencies.

The first of these is, that they give different final equations, according to the different combinations which they are made to undergo, in order to reduce them to a smaller number of equations.

Thus, for example, if there be three given equations, and the first be combined with the second, for the purpose of exterminating one of the unknown quantities, and then the first with the third, in order to exterminate the same un-

known quantity, we shall not have the same final equation, as would be found in combining the first with the second, and the second with the third, or the first with the third, and the third with the second.

They also, commonly, give a final equation of a much higher degree than is necessary for containing such values only as properly belong to the question.

If we have, for instance, four equations of the second degree, the successive elimination of the unknown quantities gives a final equation of the two hundred and fifty-sixth degree, whereas it ought not to exceed the sixteenth degree; and if the equations were of the third degree, the difference would be much greater.

These circumstances, and various others, arising from the equations being in some cases incomplete, led M. Bezout to reconsider this curious subject; and he accordingly gave, in his learned treatise before mentioned, a theory of the doctrine of elimination, which is free from most of the objections that had hitherto been made against it; showing, among other matters of importance, in this branch of analysis, that the degree of the final equation, resulting from the elimination of any number of complete equations, containing a like number of unknown quantities, and of any degrees whatever, is equal to the product of their exponents; but the subject, from the general and abstract manner in which he has treated it, is too intricate to admit of any farther details.

APPENDIX.

OF THE APPLICATION OF ALGEBRA TO GEOMETRY.

IN the preceding part of the work, we have considered Algebra as an independent science, and demonstrated its rules and operations from its own principles; but as the numerous applications of which it is susceptible ought not to be wholly overlooked, we shall now show its use in the resolution of geometrical problems, and the investigations of such theorems as depend upon the properties of right lines and the circle; which constitute what is usually called common, or plane Geometry (*a*).

For this purpose it may be observed, that when any proposition of the kind here mentioned is required to be resolved algebraically, it will be necessary, in the first place, to draw a figure that shall represent the several parts, or conditions, of

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(*a*) The learner, before he can obtain a competent knowledge of the method of application above mentioned, must first make himself master of the principal propositions of Euclid, or of those contained in my Elements of Geometry, where he will find all the essential principles of the science comprised within a much shorter compass.

And in such cases where it may be requisite to extend this mode of application to trigonometry, mechanics, or any other branch of mathematics, a previous knowledge of the nature and principles of these subjects will be equally necessary.

the problem under consideration, and to regard it as the true one.

Then, having properly considered the nature of the question, the figure so formed, must, if necessary, be still farther prepared for solution, by producing, or drawing, such lines in it as may appear, by their connexion or relations to each other, to be most conducive to the end proposed.

This being done, let the unknown line, or lines, which it is judged will be the easiest to find, together with those that are known, be denoted by the common algebraical symbols, or letters; then, by means of the proper geometrical theorems, make out as many equations, independent of each other, as there are unknown quantities employed; and the resolution of these, in the usual manner, will give the solution of the problem.

But as no general rules can be laid down for drawing the lines here mentioned, and selecting the properest quantities to substitute for, so as to bring out the most simple conclusions, the best means of obtaining experience in these matters will be to try the solution of the same problem in different ways; and then to apply that which succeeds the best to other cases of the same kind, when they afterwards occur.

The following directions, however, which are extracted, with some alterations, from Newton's Universal Arithmetic, and the fourth edition of Simpson's Algebra, 1775, will often be found of considerable use to the learner, by showing him how to proceed in many cases of this kind, where he would otherwise be left to his own judgment.

1st. In preparing the figure in the manner above mentioned, by producing, or drawing certain lines, let them be either parallel or perpendicular to some other lines in it, or be so drawn as to form similar triangles; and, if an angle be given, let the perpendicular be drawn opposite to it, and so as to fall, if possible, from one end of a given line.

2d. In selecting the proper quantities to substitute for, let those be chosen, whether required or not, that are nearest to the known or given parts of the figure, and by means of which the next adjacent parts may be obtained by addition or subtraction only, without using surds.

3d. When, in any problem, there are two lines, or quantities, alike related to other parts of the figure, or problem, the best way is not to make use of either of them separately, but to substitute for their sum, difference, or rectangle, or the sum of their alternate quotients; or for some other line or lines in the figure, to which they have both the same relation.

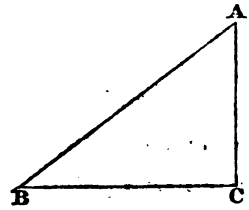
4th. When the area, or the perimeter, of a figure is given, or such parts of it as have only a remote relation to the parts that are to be found, it will sometimes be of use to assume another figure similar to the proposed one, that shall have one of its sides equal to unity, or to some other known quantity; as the other parts of the figure, in such cases, may then be determined by the known proportions of their like sides, or parts; and thence the resulting equation required.

These being the most general observations that

have hitherto been collected upon this subject, I shall now proceed to elucidate them by proper examples; leaving such farther remarks as may arise out of the mode of proceeding here used, to be applied by the learner, as occasion requires, to the solutions of the miscellaneous problems given at the end of the present article.

PROBLEM I.

The base, and the sum of the hypothenuse and perpendicular of a right angled triangle being given, it is required to determine the triangle.



Let ABC, right angled at C, be the proposed triangle; and put $BC = b$ and $AC = x$.

Then, if the sum of AB and AC be represented by s, the hypothenuse AB will be expressed by $s - x$.

But, by the well known property of right angled triangles (Euc. I, 47)

$$AC^2 + BC^2 = AB^2, \text{ or}$$
$$x^2 + b^2 = s^2 - 2sx + x^2.$$

Whence, omitting x^2, which is common to both sides of the equation, and transposing the other terms, we shall have

$$2sx = s^2 - b^2, \text{ or}$$
$$x = \frac{s^2 - b^2}{2s}, \quad . \quad . \quad . \quad . \quad (b)$$

(b) The edition of Euclid, referred to in this and all the following problems, is that of Dr. Simson, London, 1801; which

which is the value of the perpendicular AC; where s and b may be any numbers whatever, provided s be greater than b.

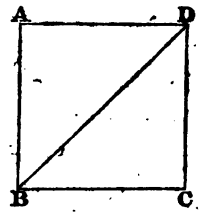
In like manner, if the base and the difference between the hypothenuse and perpendicular be given, we shall have, by putting x for the perpendicular and $d+x$ for the hypothenuse,

$$x^2+2dx+d^2=b^2+x^2, \text{ or}$$
$$x=\frac{b^2-d^2}{2d}.$$

Where the base (b) and the given difference (d) may be any numbers, as before, provided b be greater than d.

PROBLEM II.

The difference between the diagonal of a square and one of its sides being given, to determine the square



Let AC be the proposed square, and put the side BC, or CD, $=x$.

Then, if the difference of BD and BC be put $=d$, the hypothenuse BD will be $=x+d$.

But since, as in the former problem, BC^2+CD^2, or $2BC^2=BD^2$, we shall have

may also be used in the geometrical construction of these problems, should the student be inclined to exercise his talents upon this elegant, but more difficult branch of the subject.

$$2x^2 = x^2 + 2dx + d^2, \text{ or}$$
$$x^2 - 2dx = d^2.$$

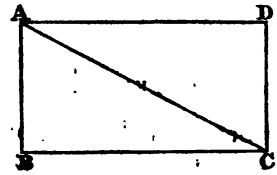
Which equation, being resolved according to the rule laid down for quadratics, in Vol. I, Art. (P), p. 124, gives

$$x = d + d\sqrt{2}.$$

Which is the value of the side BC, as was required.

PROBLEM III.

The diagonal of a rectangle ABCD, and the perimeter, or sum of all its four sides, being given, to find the sides



Let the diagonal AC $= d$, half the perimeter AB + BC $= a$, and the base BC $= x$; then will the altitude AB $= a - x$.

And since, as in the former problem, AB^2 + BC^2 = AC^2, we shall have

$$a^2 - 2ax + x^2 + x^2 = d^2, \text{ or}$$
$$x^2 - ax = \frac{d^2 - a^2}{2}.$$

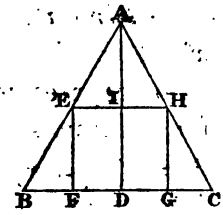
Which last equation, being resolved, as in the former instance, gives

$$x = \tfrac{1}{2}a \pm \tfrac{1}{2}\sqrt{(2d^2 - a^2)}.$$

Where a must be taken greater than d and less than $d\sqrt{2}$.

PROBLEM IV.

The base and perpendicular of any plane triangle ABC being given, to find the side of its inscribed square.



Let EG be the inscribed square; and put BC $= b$, AD $= p$, and the side of the square EH or EF $= x$.

Then, because the triangles ABC, AEH, are similar, (Euc. VI, 4,) we shall have

$$\text{AD} : \text{BC} :: \text{AI} : \text{EH}, \text{ or}$$

$$p : b :: (p-x) : x.$$

Whence, taking the products of the means and extremes, there will arise

$$px = bp - bx.$$

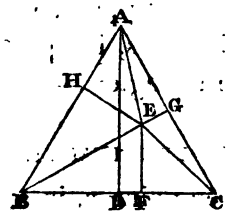
Which, by transposition and division, gives

$$x = \frac{bp}{b+p}.$$

Where b and p may be any numbers whatever.

PROBLEM V.

Having the lengths of three perpendiculars, EF, EG, EH, drawn from a certain point E, within an equilateral triangle ABC, to its three sides, to determine the sides.



Draw the perpendicular AD, and having joined

EA, EB, and EC, put EF $= a$, EG $= b$, EH $= c$, and BD (which is $\frac{1}{2}$BC) $= x$.

Then, since AB, BC, or CA, are each $= 2x$, we shall have, by Euc. I, 47,

$$\text{AD} = \sqrt{(\text{AB}^2 - \text{BD}^2)} = \sqrt{(4x^2 - x^2)} = \sqrt{3x^2} = x\sqrt{3}.$$

And because the area of any plane triangle is equal to half the rectangle of its base and perpendicular, it follows, that

$$\triangle \text{ABC} = \tfrac{1}{2}\text{BC} \times \text{AD} = x \times x\sqrt{3} = x^2\sqrt{3},$$
$$\triangle \text{BEC} = \tfrac{1}{2}\text{BC} \times \text{EF} = x \times a \quad = ax,$$
$$\triangle \text{AEC} = \tfrac{1}{2}\text{AC} \times \text{EG} = x \times b \quad = bx,$$
$$\triangle \text{AEB} = \tfrac{1}{2}\text{AB} \times \text{EH} = x \times c \quad = cx.$$

But the last three triangles, BEC, AEC, AEB, are, together, equal to the whole triangle ABC; whence

$$x^2\sqrt{3} = ax + bx + cx.$$

And, consequently, if each side of this equation be divided by x, we shall have

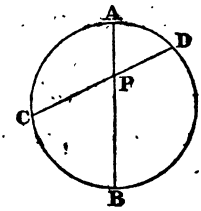
$$x\sqrt{3} = a + b + c, \text{ or}$$
$$x = \frac{a + b + c}{\sqrt{3}}.$$

Which is, therefore, half the length of either of the three equal sides of the triangle.

Cor. Since, from what is above shown, AD is $= x\sqrt{3}$, it follows, that the sum of all the perpendiculars, drawn from any point in an equilateral triangle to each of its sides, is equal to the whole perpendicular of the triangle.

PROBLEM VI.

Through a given point P, in a given circle ACBD, to draw a cord CD, of a given length.



Draw the diameter APB; and put CD $= a$, AP $= b$, PB $= c$, and CP $= x$; then will PD $= a - x$.

But, by the property of the circle, (Euc. III, 35,), CP × PD = AP × PB; whence

$$x(a-x) = bc, \text{ or}$$
$$x^2 - ax = -bc.$$

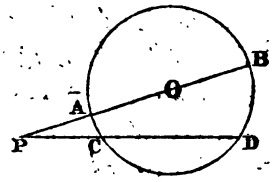
Which equation, being resolved in the usual way, gives

$$x = \tfrac{1}{2}a \pm \sqrt{(\tfrac{1}{4}a^2 - bc)};$$

where x has two values, both of which are positive.

PROBLEM VII.

Through a given point P, without a given circle ABDC, to draw a right line so that the part CD, intercepted by the circumference, shall be of a given length.



Draw PAB through the center O; and put CD $= a$, PA $= b$, PB $= c$, and PC $= x$; then will PD $= x + a$.

But, by the property of the circle, (Euc. III, 36, cor.,) PC × PD = PA × PB; whence

$$x(x+a) = bc, \text{ or}$$
$$x^2 + ax = bc.$$

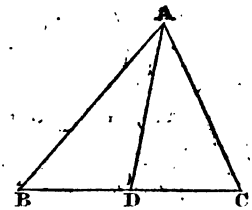
Which equation being resolved, as in the former problem, gives

$$x = -\tfrac{1}{2}a \pm \sqrt{(\tfrac{1}{4}a^2 + bc)};$$

where one value of x is positive and the other negative (*c*).

PROBLEM VIII.

The base BC, of any plane triangle ABC, the sum of the sides AB, AC, and the line AD, drawn from the vertex to the middle of the base, being given, to determine the triangle.



Put BD or DC $= a$, AD $= b$, AB + AC $= s$, and AB $= x$; then will AC $= s - x$.

But, by my Geometry, B. II, 19, $AB^2 + AC^2 = 2BD^2 + 2AD^2$; whence

$$x^2 + (s - x)^2 = 2a^2 + 2b^2, \text{ or}$$
$$x^2 - sx = a^2 + b^2 - \tfrac{1}{2}s^2.$$

Which last equation, being resolved as in the former instances, gives

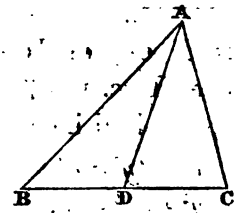
$$x = \tfrac{1}{2}s \pm \sqrt{(a^2 + b^2 - \tfrac{1}{4}s^2)},$$

(*c*) The two last problems, with a few slight alterations, may be readily employed for finding the roots of quadratic equations by construction; but this, as well as the methods frequently given for constructing some of the higher orders of equations, is a matter of little importance in the present state of mathematical science.

for the values of the two sides AB and AC of the triangle, taking the sign + for one of them, and − for the other.

PROBLEM IX.

The two sides AB, AC, and the line AD, bisecting the vertical angle, of any plane triangle, ABC, being given, to find the base BC.



Put AB $= a$, AC $= b$, AD $= c$, and BC $= x$; then, by Euc. VI, 3, we shall have

$$\text{AB}(a) : \text{AC}(b) :: \text{BD} : \text{DC}.$$

And, consequently, by the composition of ratios (Euc. V, 18,)

$$a+b : a :: x : \text{BD} = \frac{ax}{a+b}, \text{ and}$$

$$a+b : b :: x : \text{DC} = \frac{bx}{a+b}.$$

But, by Euc. VI, 13, BD × DC + AD² = AB × AC; wherefore, also,

$$\frac{abx^2}{(a+b)^2} + c^2 = ab, \text{ or}$$

$$abx^2 = (a+b)^2 \times (ab - c^2).$$

From which last equation we have

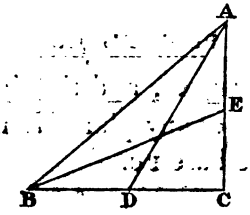
$$x = (a+b)\sqrt{\frac{ab-c^2}{ab}};$$

which is the value of the base BC, as required.

PROBLEM X.

Having given the lengths of two lines, AD, BE,

drawn from the acute angles of a right angled triangle ABC, to the middle of the opposite sides, it is required to determine the triangle.



Put AD $=a$, BE $=b$, CD, or $\frac{1}{2}$CB $=x$, and CE or $\frac{1}{2}$CA $=y$; then, since (Euc. I, 47) CD^2 + CA^2 = AD^2, and CE^2 + CB^2 = BE^2, we shall have

$$x^2+4y^2=a^2,$$
$$y^2+4x^2=b^2.$$

Whence, taking the second of these equations from four times the first, there will arise

$$15y^2=4a^2-b^2, \text{ or}$$
$$y=\sqrt{\frac{4a^2-b^2}{15}}.$$

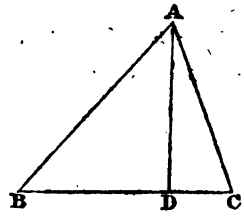
And, in like manner, taking the first of the same equations from four times the second, there will arise

$$15x^2=4b^2-a^2, \text{ or}$$
$$x=\sqrt{\frac{4b^2-a^2}{15}}.$$

Which values of x and y are half the lengths of the base and perpendicular of the triangle.

PROBLEM XI.

Having given the ratio of the two sides of a plane triangle ABC, and the segments of the base, made by a perpendicular falling from the vertical angle, to determine the triangle.



Put BD $=a$, DC $=b$, AB $=x$, AC $=y$, and the ratio of the sides as m to n.

Then, since, by the question, AB : AC :: m : n, and by B. II, 16, of my Geometry, $AB^2-AC^2=BD^2-DC^2$, we shall have

$$x : y :: m : n, \text{ and}$$
$$x^2-y^2=a^2-b^2.$$

But, by the first of these expressions, $nx=my$, or $y=\frac{nx}{m}$; whence, if this be substituted for y in the second, there will arise

$$x^2-\frac{n^2}{m^2}x^2=a^2-b^2, \text{ or}$$
$$(m^2-n^2)x^2=m^2(a^2-b^2).$$

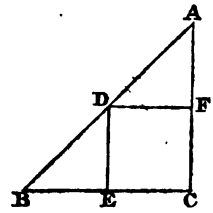
And, consequently, by division and extracting the square root, we shall have

$$x=m\sqrt{\frac{a^2-b^2}{m^2-n^2}}, \text{ and}$$
$$y=n\sqrt{\frac{a^2-b^2}{m^2-n^2}};$$

which are the values of the two sides AB, AC, of the triangle, as was required.

PROBLEM XII.

Given the hypothenuse of a right angled triangle ABC, and the side of its inscribed square DC, to find the other two sides of the triangle.



Put AB $=h$, DE, or DF $=s$, AC $=x$, and CB $=y$; then, by similar triangles, we shall have

$$\text{AC}(x) : \text{CB}(y) :: \text{AF}(x-s) : \text{FD}(s).$$

And, consequently, by multiplying the means and extremes,

$$xy - sy = sx, \text{ or}$$
$$xy = s(x+y), \quad . \quad . \quad . \quad . \quad (1)$$

But since, by Euc. I, 47, $\text{AC}^2 + \text{CB}^2 = \text{AB}^2$, we shall likewise have

$$x^2 + y^2 = h^2. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Whence, if twice equation (1) be added to equation (2), there will arise

$$x^2 + 2xy + y^2 = h^2 + 2s(x+y), \text{ or}$$
$$(x+y)^2 - 2s(x+y) = h^2.$$

Which equation, being resolved after the manner of a quadratic, gives

$$x + y = s \pm \sqrt{(h^2 + s^2)}, \text{ or}$$
$$y = s - x \pm \sqrt{(h^2 + s^2)}.$$

Hence, if this value be substituted for y in equation (1), there will arise

$$x\{s - x \pm \sqrt{(h^2 + s^2)}\} = s\{s \pm \sqrt{(h^2 + s^2)}\}, \text{ or}$$
$$x^2 - \{s \pm \sqrt{(h^2 + s^2)}\}x = -s\{s \pm \sqrt{(h^2 + s^2)}\}.$$

And, consequently, by resolving this last equation, we shall have

$$x = \tfrac{1}{2}\{s \pm \sqrt{(h^2+s^2)}\} \pm \sqrt{\{\tfrac{1}{4}h^2 - \tfrac{1}{2}s^2 \mp \tfrac{s}{2}\sqrt{(h^2+s^2)}\}}$$

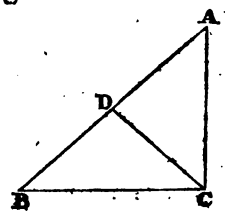
and

$$y = \tfrac{1}{2}\{s \pm \sqrt{(h^2+s^2)}\} \mp \sqrt{\{\tfrac{1}{4}h^2 - \tfrac{1}{2}s^2 \mp \tfrac{s}{2}\sqrt{(h^2+s^2)}\}}$$

Which are the values of the perpendicular AC and base BC, as was required.

PROBLEM XIII.

Having given the perimeter of a right angled triangle ABC, and the perpendicular CD, falling from the right angle on the hypothenuse, to determine the triangle.



Put p = perimeter, CD $= a$, AC $= x$, and BC $= y$; then AB $= p - (x + y)$.

But, by right angled triangles (Euc. I, 47) $AC^2 + BC^2 = AB^2$; whence

$$x^2 + y^2 = p^2 - 2p(x+y) + x^2 + 2xy + y^2.$$

Or, by transposing the terms and dividing by 2,

$$p(x+y) - \tfrac{1}{2}p^2 = xy. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

And since, by similar triangles, AB : BC :: AC : CD, we shall also have, by multiplying the means and extremes,

$$AB \times CD = BC \times AC, \text{ or}$$

$$ap - a(x+y) = xy. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Whence, by comparing equation (1) with equation (2), there will arise

$$(a+p) \times (x+y) = ap + \tfrac{1}{2}p^2.$$

Where

$$x+y=\frac{p(a+\frac{1}{2}p)}{a+p}, \text{ or}$$

$$y=\frac{p(a+\frac{1}{2}p)}{a+p}-x.$$

And, if these values be now substituted for $x+y$ and y in equation (2), the result, when simplified and reduced, will give

$$(a+p)x^2-p(a+\tfrac{1}{2}p)x=-\tfrac{1}{2}ap^2.$$

From which last equation and the value of y, above found, we shall have

$$x \text{ or AC}=\frac{p(a+\frac{1}{2}p)}{2(a+p)}\pm\frac{p}{2(a+p)}\sqrt{\{(a-\tfrac{1}{2}p)^2-2a^2\}}$$

and

$$y \text{ or BC}=\frac{p(a+\frac{1}{2}p)}{2(a+p)}\mp\frac{p}{2(a+p)}\sqrt{\{(a-\tfrac{1}{2}p)^2-2a^2\}}$$

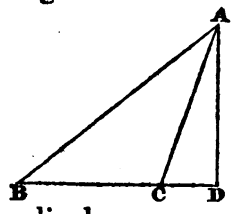
And, if the sum of these two sides be taken from p, the result will give

$$\text{AB}=p-(x+y)=\frac{p^2}{2(a+p)}.$$

Which expressions are, therefore, respectively equal to the values of the three sides of the triangle.

PROBLEM XIV.

Given the perpendicular, base, and sum of the sides of an obtuse angled plane triangle ABC; to determine the triangle.



Let the perpendicular AD $=p$, the base BC $=b$, the sum of AB and AC $=s$, and their difference $=x$.

Then, since half the difference of any two quantities added to half their sum gives the greater, and, when subtracted, the less, we shall have

$$AB = \tfrac{1}{2}(s+x), \text{ and } AC = \tfrac{1}{2}(s-x).$$

But, by Euc. I, 47, $CD^2 = AC^2 - AD^2$, or $CD = \sqrt{\{\tfrac{1}{4}(s-x)^2 - p^2\}}$; and, by B. II, 12, $AB^2 = BC^2 + AC^2 + 2BC \times CD$; whence

$$\tfrac{1}{4}(s+x)^2 = b^2 + \tfrac{1}{4}(s-x)^2 + 2b\sqrt{\{\tfrac{1}{4}(s-x)^2 - p^2\}}, \text{ or}$$
$$sx - b^2 = 2b\sqrt{\{\tfrac{1}{4}(s-x)^2 - p^2\}}.$$

And if each of the sides of this last equation be squared, there will arise, by transposition and simplifying the result,

$$(s^2 - b^2)x^2 = b^2(s^2 - b^2) - 4b^2p^2, \text{ or}$$
$$x = b\sqrt{(1 - \frac{4p^2}{s^2 - b^2})}.$$

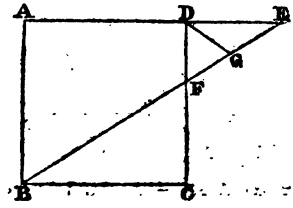
Whence, by addition and subtraction, we shall have

$$AB = \frac{s}{2} + \frac{b}{2}\sqrt{(1 - \frac{4p^2}{s^2 - b^2})}, \text{ and}$$
$$AC = \frac{s}{2} - \frac{b}{2}\sqrt{(1 - \frac{4p^2}{s^2 - b^2})}.$$

Which are the sides of the triangle, as was required.

PROBLEM XV.

It is required to draw a right line BFE from one of the angles B of a given square BD, so that the part FE, intercepted by DE and DC, shall be of a given length.



Bisect FE in G, and put AB or BC $= a$, FG or GE $= b$, and BG $= x$; then will BE $= x + b$ and BF $= x - b$.

But since, by right angled triangles, $AE^2 = BE^2 - AB^2$, we shall have

$$AE = \sqrt{\{(x+b)^2 - a^2\}}.$$

And, because the triangles BCF, EAB, are similar,

$$BF : BC :: BE : AE, \text{ or}$$

$$a(x+b) = (x-b)\sqrt{\{(x+b)^2 - a^2\}}.$$

Whence, by squaring each side of this equation, and arranging the terms in order, there will arise

$$x^4 - 2(a^2 + b^2)x^2 = b^2(2a^2 - b^2).$$

Which equation, being resolved after the manner of a quadratic, will give

$$x = \sqrt{\{a^2 + b^2 \pm a\sqrt{(a^2 + 4b^2)}\}}.$$

And, consequently, by adding b to, or subtracting it from this last expression, we shall have

$$BE = \sqrt{\{a^2 + b^2 \pm a\sqrt{(a^2 + 4b^2)}\}} + b, \text{ or}$$

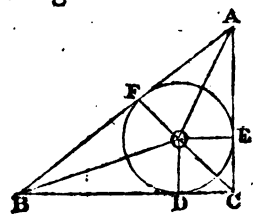
$$BF = \sqrt{\{a^2 + b^2 \pm a\sqrt{(a^2 + 4b^2)}\}} - b.$$

Which values, by determining the point E, or F, will satisfy the problem.

Where it may be observed, that the point G lies in the circumference of a circle, described from the centre D, with the radius FG.

PROBLEM XVI.

The perimeter of a right angled triangle ABC, and the radius of its inscribed circle being given, to determine the triangle.



Let the perimeter of the triangle $=p$, the radius OD, OE, or OF of the inscribed circle $=r$, AE $=x$, and BD $=y$.

Then, since in the right angled triangles AEO, AFO, OE is equal to OF, and AO is common, AF will also be equal AE, or x.

And, in like manner, it may be shown, that BF is equal to BD, or y.

But, by the question, and Euc. I, 47, we have

$$(x+r)+(y+r)+(x+y)=p, \text{ and}$$
$$(x+r)^2+(y+r)^2=(x+y)^2.$$

Or, by adding the terms of the first, and squaring those of the second,

$$x+y=\tfrac{1}{2}p-r, \text{ and}$$
$$r(x+y)=xy-r^2.$$

Hence, since, in the first of these equations, $y=(\frac{1}{2}p-r)-x$, if this value be substituted for y in the second, there will arise

$$x^2-(\tfrac{1}{2}p-r)x=-r(\tfrac{1}{2}p-r).$$

Which equation, being resolved in the usual manner, gives

$$x=\tfrac{1}{2}(\tfrac{1}{2}p-r)\pm\sqrt{\{\tfrac{1}{4}(\tfrac{1}{2}p-r)^2-r(\tfrac{1}{2}p-r)\}},$$
and
$$y=\tfrac{1}{2}(\tfrac{1}{2}p-r)\mp\sqrt{\{\tfrac{1}{4}(\tfrac{1}{2}p-r)^2-r(\tfrac{1}{2}p-r)\}}.$$

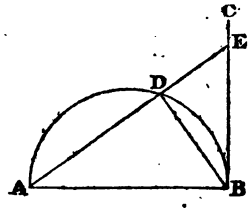
And, consequently, if r be added to each of these last expressions, we shall have

$$\text{AC}=\tfrac{1}{2}(\tfrac{1}{2}p+r)\pm\sqrt{\{\tfrac{1}{4}(\tfrac{1}{2}p-r)^2-r(\tfrac{1}{2}p-r)\}},$$
and
$$\text{BC}=\tfrac{1}{2}(\tfrac{1}{2}p+r)\mp\sqrt{\{\tfrac{1}{4}(\tfrac{1}{2}p-r)^2-r(\tfrac{1}{2}p-r)\}},$$

for the values of the perpendicular and base of the triangle, as was required.

PROBLEM XVII.

From one of the extremities A, of the diameter of a given semicircle ADB, to draw a right line AE, so that the part DE, intercepted by the circumference and a perpendicular drawn from the other extremity, shall be of a given length.



Let the diameter AB $= d$, DE $= a$, and AE $= x$; and join BD.

Then, because the angle ADB is a right angle, (Euc. III, 31,) the triangles ABE, ABD, are similar.

And, consequently, by comparing their like sides, we shall have

$$AE : AB :: AB : AD, \text{ or}$$
$$x : d :: d : x - a.$$

Whence, multiplying the means and extremes of these proportionals, there will arise

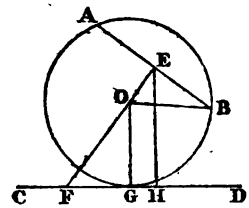
$$x^2 - ax = d^2.$$

Which equation, being resolved after the usual manner, will give

$$x = \tfrac{1}{2}a + \sqrt{(\tfrac{1}{4}a^2 + d^2)}.$$

PROBLEM XVIII.

To describe a circle through two given points A, B, that shall touch a right line CD given in position.



Join AB; and through O, the assumed centre of the required circle, draw FE perpendicular to AB; which will bisect it in E (Euc. III, 3).

Also, join OB; and draw EH, OG, perpendicular to CD; the latter of which will fall on the point of contact G (Euc. III, 18).

Hence, since A, E, B, H, F, are given points, put $EB = a$, $EF = b$, $EH = c$, and $EO = x$; which will give $OF = b - x$.

Then, because the triangle OEB is right angled at E, we shall have

$$OB^2 = EO^2 + EB^2, \text{ or}$$
$$OB = \sqrt{(x^2 + a^2)}.$$

But, by similar triangles, FE : EH :: FO : OG, or OB; or $b : c :: b - x :$ OB; whence, also,

$$OB = \frac{c}{b}(b - x).$$

And, consequently, if these two values of OB be put equal to each other, there will arise

$$\sqrt{(x^2 + a^2)} = \frac{c}{b}(b - x).$$

Or, by squaring each side of this equation, and simplifying the result,

$$(b^2 - c^2)x^2 + 2bc^2x = b^2(c^2 - a^2).$$

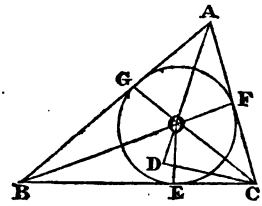
Which last equation, when resolved in the usual manner, gives

$$x = -\frac{bc^2}{b^2-c^2} + b\sqrt{\left\{\frac{c^4}{(b^2-c^2)^2} + \frac{c^2-a^2}{b^2-c^2}\right\}},$$

for the distance of the centre O from the chord AB.*

PROBLEM XIX.

The three lines AO, BO, CO, drawn from the angular points of a plane triangle ABC, to the centre of its inscibed circle, being given, to find the radius of the circle, and the sides of the triangle.



Let O be the centre of the circle, and, on AO produced, let fall the perpendiculars CD; and draw OE, OF, OG, to the points of contact E, F, G.

Then, because the three angles of the triangle ABC are, together, equal to two right angles, (Euc. I, 32,) the sum of their halves OAC + OCA + OBE will be equal to one right angle.

But the sum of the two former of these, OAC + OCA, is equal to the external angle DOC; whence the sum of DOC + OBE, as also of DOC + OCD, is equal to a right angle; and, consequently, OBE = OCD.

Let, therefore, AO = a, BO = b, CO = c, and the radius OE, OF, or OG = x.

Then, since the triangles BOE, COD are similar, BO : OE :: CO : OD, or $b : x :: c :$ OD; which gives

$$\text{OD} = \frac{cx}{b}, \text{ and } \text{CD} = \sqrt{\left(c^2 - \frac{c^2x^2}{b^2}\right)}, \text{ or } \frac{c}{b}\sqrt{(b^2-x^2)}.$$

Also, because the triangle AOC is obtuse angled at O, we shall have (Euc. II, 12)

$$AC^2 = AO^2 + CO^2 + 2AO \times OD; \text{ or}$$

$$AC = \sqrt{(a^2 + c^2 + \frac{2acx}{b})}, \text{ or } \sqrt{(\frac{b(a^2 + c^2) + 2acx}{b})}.$$

But the triangles ACD, AOF, being likewise similar,

$$AC : CD :: AO : OF, \text{ or}$$

$$\sqrt{(\frac{b(a^2 + c^2) + 2acx}{b})} : \frac{c}{b}\sqrt{(b^2 - x^2)} :: a : x.$$

Whence, multiplying the means and extremes, and squaring the result, there will arise

$$bx^2\{b(a^2 + c^2) + 2acx\} = a^2c^2(b^2 - x^2).$$

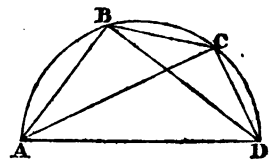
Or, by collecting the terms together, and dividing by the coefficient of the highest power of x,

$$x^3 + (\frac{ab}{2c} + \frac{ac}{2b} + \frac{bc}{2a})x^2 = \frac{abc}{2}.$$

From which last equation x may be determined, and thence the sides of the triangle (*d*).

PROBLEM XX.

Given the three sides AB, BC, CD, of a trapezium ABCD, inscribed in a semicircle, to find the diameter, or remaining side AD.



(*d*) This, and the following problem, cannot be constructed geometrically, or by means only of right lines and the circle, being what the ancients usually denominated solid problems, from the circumstance of their involving an equation of more than two dimensions.

Let AB $= a$, BC $= b$, CD $= c$, and AD $= x$; then, by Euc. VI, D, AC × BD = AD × BC + AB × CD, or $= bx + ac$.

But ABD, ACD, being right angles, (Euc. III, 31,) we shall have

$$AC = \sqrt{(AD^2 - DC^2)}, \text{ or } \sqrt{(x^2 - c^2)}, \text{ and}$$
$$BD = \sqrt{(AD^2 - AB^2)}, \text{ or } \sqrt{(x^2 - a^2)}.$$

Whence, by substituting these two values in the former expression, there will arise

$$\sqrt{(x^2 - c^2)} \times \sqrt{(x^2 - a^2)} = bx + ac.$$

Or, by squaring each side, and reducing the result,

$$x^3 - (a^2 + b^2 + c^2)x = 2abc.$$

From which last equation, the value of x may be found, as in the last problem (*e*).

MISCELLANEOUS PROBLEMS.

PROBLEM I.

To find the side of a square, inscribed in a given semicircle, whose diameter is d.

Ans. $\frac{1}{5}d\sqrt{5}$

PROBLEM II.

Having given the hypothenuse (13) of a right

(*e*) Newton, in his Universal Arithmetic, English edition, 1728, has resolved this problem in a variety of different ways, in order to show, that some methods of proceeding, in cases of this kind, frequently lead to more elegant solutions than others; and that a ready knowledge of these can only be obtained by practice and observation.

angled triangle, and the difference between the other two sides (7), to find these sides(*f*).

Ans. 5 and 12

PROBLEM III.

To find the side of an equilateral triangle, inscribed in a circle, whose diameter is d; and that of another circumscribed about the same circle.

Ans. $\frac{1}{2}d\sqrt{3}$, and $d\sqrt{3}$

PROBLEM IV.

To find the side of a regular pentagon, inscribed in a circle, whose diameter is d.

Ans. $\frac{1}{4}d\sqrt{(10-2\sqrt{5})}$

PROBLEM V.

To find the sides of a rectangle, the perimeter of which shall be equal to that of a square, whose side is a, and its area half that of the square.

Ans. $a+\frac{1}{2}a\sqrt{2}$ and $a-\frac{1}{2}a\sqrt{2}$

PROBLEM VI.

Having given the side (10) of an equilateral triangle, to find the radii of its inscribed and circumscribing circles. Ans. 2.8868 and 5.7736

PROBLEM VII.

Having given the perimeter (12) of a rhombus, and the sum (8) of its two diagonals, to find the diagonals. Ans. $4+\sqrt{2}$ and $4-\sqrt{2}$

(*f*) Such of these questions as are proposed in numbers, should first be resolved generally, by means of the usual symbols, and then reduced to the answers above given, by substituting the numeral values of the letters in the results thus obtained.

PROBLEM VIII.

Required the area of a right angled triangle, whose hypothenuse is x^{3x}, and the base and perpendicular x^{2x} and x^{x}. Ans. 1.029085

PROBLEM IX.

Having given the two contiguous sides (a, b) of a parallelogram, and one of its diagonals (d), to find the other diagonal. Ans. $\sqrt{(2a^2+2b^2-d^2)}$

PROBLEM X.

Having given the perpendicular (300) of a plane triangle, the sum of the two sides (1150), and the difference of the segments of the base (495), to find the base and the sides.

Ans. 945, 375, and 780

PROBLEM XI.

The lengths of three lines drawn from the three angles of a plane triangle to the middle of the opposite sides, being 18, 24, and 30, respectively; it is required to find the sides.

Ans. 20, 28.844, and 34.176

PROBLEM XII.

In a plane triangle, there is given the base (50), the area (796), and the difference of the sides (10), to find the sides and the perpendicular.

Ans. 36, 46, and 33.261

PROBLEM XIII.

Given the base (194) of a plane triangle, the line that bisects the vertical angle (66), and the diameter (200) of the circumscribing circle, to find the other two sides.

Ans. 81.36587 and 157.43865

PROBLEM XIV.

The lengths of two lines that bisect the acute angles of a right angled plane triangle, being 40 and 50 respectively, it is required to determine the three sides of the triangle.

Ans. 35.80737, 47.40728, and 59.41143

PROBLEM XV.

Given the altitude (4), the base (8), and the sum of the sides (12), of a plane triangle, to find the sides.

Ans. $6+\frac{4}{5}\sqrt{5}$ and $6-\frac{4}{5}\sqrt{5}$

PROBLEM XVI.

Having given the base of a plane triangle (15), its area (45), and the ratio of its other two sides as 2 to 3, it is required to determine the lengths of these sides.

Ans. 7.7915 and 11.6872

PROBLEM XVII.

Given the perpendicular (24), the line bisecting the base (40), and the line bisecting the vertical angle (25), to determine the triangle.

Ans. The base $\frac{250}{7}\sqrt{7}$

PROBLEM XVIII.

Given the hypothenuse (10) of a right angled triangle, and the difference of two lines drawn from its extremities to the centre of the inscribed circle (2), to determine the base and perpendicular.

Ans. 8.08004 and 5.87447

PROBLEM XIX.

Having given the lengths (a, b,) of two chords, cutting each other at right angles, in a circle, and the distance (c) of their point of intersection from the centre, to determine the diameter of the circle.

Ans. $\sqrt{\{8(a^2+b^2)+2c^2\}}$

PROBLEM XX.

Two trees, standing on an horizontal plane, are 120 feet asunder; the height of the highest of which is 100 feet, and that of the shortest 80; whereabouts in the plane must a person place himself, so that his distance from the top of each tree, and the distance of the tops themselves, shall be all equal to each other?

Ans. $20\sqrt{21}$ feet from the bottom of the shortest, and $40\sqrt{3}$ feet from the bottom of the other

PROBLEM XXI.

Having given the sides of a trapezium, inscribed in a circle, equal to 6, 4, 5, and 3, respectively, to determine the diameter of the circle.

Ans. $\frac{1}{20}\sqrt{(130\times153)}$, or 7.051595

PROBLEM XXII.

Supposing the town A to be 30 miles from B, B 25 miles from C, and C 20 miles from A; whereabouts must a house be erected that shall be at an equal distance from each of them?

Ans. 15.118556 miles from each

PROBLEM XXIII.

In a plane triangle, having given the perpen-

dicular (p); and the radii (r, R) of its inscribed and circumscribing circles, to determine the triangle.

Ans. The base $\frac{2r\sqrt{(2pR-4rR-r^2)}}{p-2r}$

PROBLEM XXIV.

Having given the base of a plane triangle equal to $2a$, the perpendicular equal to a, and the sum of the cubes of its other two sides equal to three times the cube of the base; to determine the sides (g).

Ans. $a(2+\frac{1}{3}\sqrt{6})$ and $a(2-\frac{1}{3}\sqrt{6})$

APPLICATION OF ALGEBRA TO THE DOCTRINE OF CURVES.

THE use of Algebra, in the resolution of problems in common Geometry, appears to have been as ancient as the first introduction of that science into Europe; several propositions, relating to plane triangles, having been resolved, in this way, by

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(g) This curious problem was first proposed in the Ladies' Diary, for the year 1794, in the following form:

In a palace of one of the Persian kings, there is said to have been a triangular figure such, that the sum of the cubes of two of its sides was equal to three times the cube of its base, or other side, which was 200 feet in length; and the area, or space, inclosed within its boundaries, contained just 10000 square feet; from which data it is required to construct the triangle by common, or plane, geometry.

This was accordingly done, in the Diary for the following year, where there was given the construction of the proposer, and two others, by different persons; for a farther account of which, as well as for several algebraical solutions of the problem, see the *Scriptores Logarithmici* of Maseres, Vol. IV, p. 335, *et seq.*

Regiomontanus, in his Treatise of Trigonometry, written about the year 1464, which was some time prior in date to the *Summa Arithmetica*, and other works, of Lucas de Burgo, that have commonly been considered as the earliest modern productions on algebra now extant.

But the application of this science to the doctrine of curves, which began to be employed at a much later period, is due to the celebrated Descartes; who, in his Geometry, published in 1637, first laid down the method, now generally followed, of expressing the properties and relations of curves by means of algebraic equations; and thus gave an impulse to this branch of science, which, from the improvements of later writers, and the subsequent invention of fluxions, has become a subject of the greatest utility and importance.

Curves, or, as they are commonly called, geometrical lines, when treated of in this manner, are divided into classes, or orders, according to the dimensions of the equation that expresses the relation between their ordinates and abscissæ; or, which amounts to the same thing, according to the number of points by which they may be cut by a right line (*h*).

(*h*) The ancients admitted into geometry only right lines and the circle, and, in some cases, the conic sections; but the moderns, in consequence of the great extent to which this branch has since been carried, have agreed to introduce into it all kinds of lines, that can be expressed by equations. They have, also, laid it down as a general rule, that no line of a superior order is to be used in the construction of a problem when it can be done by one of an inferior order; and that the curve, which is the

Thus, a line of the first order is a right line; since it can only be cut by another right line in one point, and is expressed by the simple equation,

$$y + ax + b = 0 \quad . \; . \; . \; . \; . \; . \quad (i).$$

Lines of the second order, or curves of the first kind, are the circle and the conic sections, all of which may be cut by a right line in two points, and are expressed by the equation,

$$y^2 + (ax + b)y + cx^2 + dx + e = 0.$$

Lines of the third order, or curves of the second kind, are such as can be cut by a right line in three points; their most general equation being

$$x^3 + (ax + b)y^2 + (cx^2 + dx + e)y + fx^3 + gx^2 + hx + i = 0.$$

And, in the same way, we may proceed for any other order; observing, that in all cases of this kind, x denotes the abscissa, and y its corresponding ordinate; and that a, b, c, &c. are given, or constant, quantities, affected with the signs + or −, as it may happen; of which one or more may

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most easily described, is always to be preferred, in this case, to one that is more complex, without any regard to the greater or less degree of simplicity of the equation by which it is expressed. See the appendix to Newton's Universal Arithmetic, where the reader will find a number of observations on this subject, which are worthy of his attention.

(*i*) Since this equation, which is of the first degree, comprises only the right line, to which the name of curve cannot be applied, it has been judged proper to distinguish the circle, conic sections, &c. by the generic appellation of lines, instead of that of curves, in order to render the nomenclature of this branch of science, conformable to the nature of the equations by which the several curves are expressed.

vanish, or be wanting, provided the equation, by such a defect, does not become of a lower dimension.

Curves are also distinguished by the denominations of algebraical or geometrical, and transcendental or mechanical.

Algebraical, or geometrical, curves, are those of which the relations of the abscissæ and ordinates can be expressed by a common algebraic equation; as in the circle and conic sections, above mentioned.

Transcendental, or mechanical, curves, are such as cannot be expressed by a common algebraic equation; as in the case where one or more of the terms of the equation are of the form a log. x, or a sin. y, &c., or a quantity denoting a curve line.

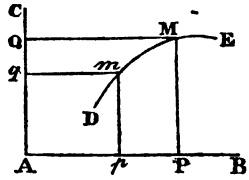
Exponential curves, are such as are expressed by equations which consist of terms whose indices are indeterminate, or variable, quantities, as

$$a^x = y, \text{ or } y^x + x^y = axy.$$

Also, if all the points of a curve lie in the same plane, it is called a plane curve, and if they lie in different planes, it is said to be a curve of double curvature.

The axis of a curve is a right line drawn from the vertex through the centre, or middle, of the curve; and if it bisects the ordinates it is called a diameter.

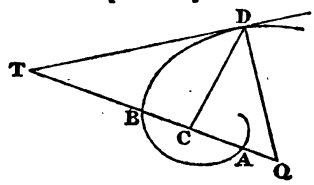
The abscissæ of a curve (DE) are any parts, or segments, (Ap, AP&c.) of the axis, or diameter (AB), which originate at the vertex, or some other fixed point (A), and are terminated by the ordinates, (pm, PM&c.;) which are lines drawn from certain points in the curve, perpendicular to the former, or making the same given angle with them.



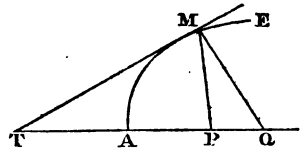
An absciss and its corresponding ordinate (as AP, PM, or AP, AQ), whether right or oblique, when spoken of together, are frequently called co-ordinates; and the fixed lines, (AB, AC,) which are supposed to be so drawn as to determine their directions, are called the axes of the co-ordinates.

Hence, agreeably to what has been before observed, if the abscissa (AP), be denoted by x, and its corresponding ordinate (PM), by y, the equation, which expresses the relation between AP and PM, or between x and y, is called the equation of the curve (DE); and, conversely, the curve is called the *locus* of that equation.

If the curve turn round a fixed point, or centre, it is called a spiral, or radial curve, as ABD; in which case a line, CD, drawn from the centre (C), to any point in the curve, is an ordinate; and the absciss is a variable quantity.



A tangent to a curve (AME), is a right line (TM), which touches the curve at a point (M), without cutting it; or so that no other right line can be drawn through the point of contact between it and the curve.

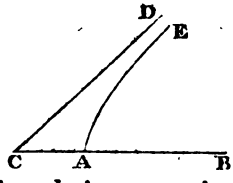


The subtangent, is the distance (PT), of the ordinate (MP), from the intersection of the tangent and absciss; and in the spiral ABD (see the former fig.), it is the line CT, which is perpendicular to the ordinate CD.

The normal is a line (MQ), drawn perpendicular to the tangent (TM), from the point of contact (M), and meeting the line of the abscissæ in Q.

The subnormal (PQ), is the distance of the ordinate (MP), from that point in the line of the abscissæ where the normal (MQ), or a perpendicular to the curve, intersects it; and in the spiral, it is the line CQ, perpendicualr to CD, and cutting the line DQ, (which is perpendiuclar to the curve,) in Q.

An asymptote to a curve (AE), is a line (CD), drawn from some fixed point C, which continually approaches towards the curve, but never meets it, unless they be both considered as extended to an infinite distance.

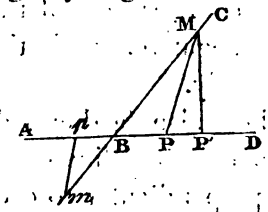


These particulars being premised, we shall now proceed to illustrate the above doctrine, by applying it chiefly to right lines, and such of the more remarkable curves, as are most likely to attract the

attention of learners; the subject being by far too copious to be treated of at length, in a work like the present (*k*).

OF THE RIGHT LINE.

1. LET there be taken, in the first place, the right lines AD, BC, cutting each other in B; and draw MP making any angle with AD.



Then, since the angles at B, P, and M, are invariable, BP will have the same constant ratio to PM, from whatever point in BC the line PM may be drawn.

Hence, putting AB $= a$, AP $= x$, PM $= y$, and the above mentioned ratio as $b : c$, we shall have

$$x - a(\text{BP}) : y(\text{PM}) :: b : c, \text{ or}$$
$$by = cx - ac.$$

Which equation expresses the relation of the right line BC to AD, when A, the origin of the abscissæ, falls beyond their point of intersection B.

And by altering the values of a, b, c, and taking x and a positive or negative, as the case may re-

(*k*) Those who are desirous of advancing beyond the elements of this branch of the science, may consult the excellent tract of Maclaurin, printed as an appendix to his Algebra; as also the second book of the *Analysis Infinitorum* of Euler, and Cramer's *Introduction à l'Analyse des Lignes Courbes*, where they will find every part of the subject clearly and ably explained.

quire, this equation may be made to coincide with any simple equation whatever.

Thus, supposing AB to be $=0$, or that the line of the abscissæ commences at B, we shall have, in that case,

$$by = cx, \text{ or } y = \frac{c}{b}x.$$

And, if the abscissa BP be considered as positive, Bp, taken on the contrary side of the point B, must be regarded as negative, and the equation be changed accordingly.

2. We may here also farther add, that it is indifferent whether the co-ordinates, in this, or any other, case, be perpendicular to each other or not.

For put $AP' = x'$, and the perpendicular ordinate $P'M = y'$; and let the angle MPP', which the oblique ordinate MP makes with AD, be denoted by φ.

Then, in the right angled triangle MP'P, we shall have

$$\sin.\varphi : \text{rad.}(1) :: y'(P'M) : y(PM) = \frac{y'}{\sin.\varphi},$$

and

$$\text{rad.}(1) : \cos.\varphi :: y(PM) : PP' = \frac{\cos.\varphi \times y'}{\sin.\varphi}.$$

Whence, since

$$x = AP = AP' - PP' = x' - \frac{\cos.\varphi \times y'}{\sin.\varphi},$$

if these values be substituted for x and y, in the expression $by = cx - ac$, before obtained, there will arise

$$\frac{by'}{\sin.\varphi} = c\left(x' - \frac{\cos.\varphi \times y'}{\sin.\varphi}\right) - ac.$$

And, consequently, by multiplication and transposition, we shall have

$$(b + c\cos.\varphi)y' = c\sin.\varphi \times x' - ac\sin.\varphi.$$

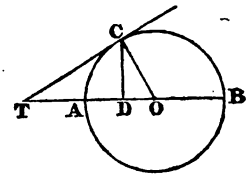
Which is evidently an equation equally general with the former.

Hence, as the perpendicularity of the ordinates to the axis does not affect the generality of the equation, we shall, for the sake of simplicity, employ them, in this way, in what follows, except when otherwise mentioned.

For the same reason, we shall, also, take the origin of the abscissæ, whenever it can be done, at the vertex, or some other remarkable point, within or without the curve; as well as pass over the transformations that are usually made of the axes of the co-ordinates, which are foreign to the design of the present article.

OF THE CIRCLE.

1. LET there be now taken the circle ACB, whose centre is O; and put the diameter AB $= d$, the abscissa AD $= x$, and the ordinate CD $= y$.



Then, by the well known property of the figure, (Euc. VI, 8,) we shall have

$$AD \times DB = CD^2, \text{ or}$$
$$dx - x^2 = y^2.$$

Which expression may be considered as the equation of the circle.

2. Or, if the radius OA, or OB, be put $= r$, and OD $= x'$, the last expression will become

$$(\text{AO}-\text{OD})(\text{AO}+\text{OD}) = \text{CD}^2, \text{ or}$$
$$r^2 - x'^2 = y^2.$$

Which is an equation of a more simple form than that above given.

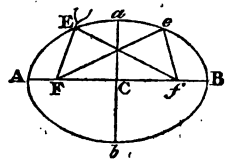
3. Also, if the right line TC be supposed to touch the circle at C, we shall have, by similar triangles,

$$\text{DO}(\text{OA}-\text{AD}) : \text{DC} :: \text{DC} : \text{DT}, \text{ or}$$
$$\text{DT} = \frac{y^2}{x'} = \frac{r^2 - x'^2}{x'} = \frac{dx - x^2}{\frac{1}{2}d - x}.$$

Which are so many different expressions for the value of the subtangent DT, or the distance of the ordinate CD from the point of intersection (T), of the tangent TC.

OF THE ELLIPSE.

1. Let a thread, or fine wire, EF*f*E, be put round two pins, fixed at the points F, *f*; then, if a pen, or pencil, be put through a knot at E, and moved round these points as centres, so as always to keep the string stretched, it will describe the curve A*a*B*b*, which is called an ellipse (*l*).

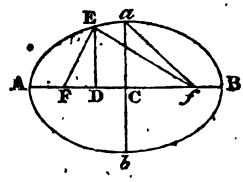


(*l*) The genesis of the ellipse, here given, affords the following ready method of describing it mechanically by points. Take as many points in the transverse axis AB as may be thought necessary; and from the foci F, *f*, with radii equal to the distances of those points from its two extremities A, B, describe a number of arcs, and their intersections will be so many points in the curve.

Where it is to be observed, that the points F, f, about which the string moves, are called the foci; and the lines AB, ab, drawn at right angles to each other through the centre C, or middle of the line Ff, are called the transverse and conjugate axes, or diameters.

2. Hence, since the string in this construction, always continues of the same length, it is evident, that the sum of two lines drawn from the foci F, f, to any point of the curve, is equal to the transverse axis AB; and that the distances FA, fB, from the vertices A, B, are also equal to each other.

3. Let, therefore, AaBb be an ellipse, described about the foci F, f, as above, and put the semi-transverse diameter AC $= t$, the semi-conjugate aC $= c$, CD $= x'$, and ED $= y$ (*m*).



Then, since, by the generation of the curve, fa is equal to $\frac{1}{2}$AB $= t$, we shall have, (Euc. I, 47,) Cf, or CF $= \sqrt{(af^2 - ac^2)} = \sqrt{(t^2 - c^2)}$; which put $= d$.

And, because the triangles EDF, EDf, are right angled at D,

(*m*) In the propositions here given, as well as in those that follow, only such properties of the different curves are noticed, as enter into the plan of the present article; which is designed chiefly as an elementary introduction to the general doctrine of curves; the student being referred for the rest to works written expressly on this subject.

$$\sqrt{(\text{FD}^2+\text{ED}^2)},\ \text{or}\ \sqrt{\{(d-x')^2+y^2\}}=\text{FE},$$

and

$$\sqrt{(f\text{D}^2+\text{ED}^2)},\ \text{or}\ \sqrt{\{(d+x')^2+y^2\}}=f\text{E};$$

and, consequently,

$$\sqrt{\{(d-x')^2+y^2\}}+\sqrt{\{(d+x')^2+y^2\}}=\text{FE}+f\text{E}=2t;\ \text{or}$$

$$\sqrt{\{(d-x')^2+y^2\}}=2t-\sqrt{\{(d+x')^2+y^2\}}.$$

Whence, if each side of this last equation be squared, there will arise

$$d^2-2dx'+x'^2+y^2=$$

$$4t^2-4t\sqrt{\{(d+x')^2+y^2\}}+d^2+2dx'+x'^2+y^2;$$

or, by transposition and division,

$$t^2+dx'=t\sqrt{\{(d+x')^2+y^2\}}.$$

Also, by again squaring this expression, there will result

$$t^4+2dt^2x'+d^2x'^2=d^2t^2+2dt^2x'+t^2x'^2+t^2y^2,\ \text{or}$$

$$t^2y^2=t^4-d^2t^2-(t^2-d^2)x'^2.$$

And, consequently, if t^2-c^2 be now substituted in the place of its equal d^2, we shall have

$$t^2y^2=t^2c^2-c^2x'^2,\ \text{or}$$

$$y^2=\frac{c^2}{t^2}(t^2-x'^2).$$

Which is the equation of the curve, when the origin of the abscissæ is taken at the centre of the ellipse.

4. And if AD be denoted by x, and $t-x$ be substituted for x', the last expression will then become

$$y^2=\frac{c^2}{t^2}(2tx-x^2).$$

Which is the equation of the curve, when the abscissa AD is taken from the end of the transverse axis AB.

5. Hence, also, if this last expression be converted into a proportion, we shall obtain, by taking the

whole diameters instead of their halves, the following proposition.

As the square of the transverse axis is to the square of the conjugate, so is the rectangle of any two abscissæ to the square of their ordinate: that is,

$$AB^2 : ab^2 :: AD \times DB : DE^{2\prime}.$$

Which theorem, together with that arising out of the description of the figure, may be considered as exhibiting the two fundamental properties of the ellipse; being those from which most of the others can be derived (*n*).

6. Again, if FE be put $=z$, CD $=x'$, and the distance CF, or Cf, of the centre from either of the foci $=d$, as before, we shall have, from what has been above shown,

$$z=\sqrt{\{(d-x')^2+y^2\}}, \text{ and}$$
$$2t-z=\sqrt{\{(d+x')^2+y^2\}}.$$

And, consequently, by squaring each of these expressions, there will arise

$$z^2=d^2-2dx'+x'^2+y^2, \text{ and}$$
$$4t^2-4tz+z^2=d^2+2dx'+x'^2+y^2.$$

(*n*) If the double ordinate in this, or either of the other conic sections, pass through the focus, it is called the *parameter* (or equal measurer); being a constant quantity, which, in the ellipse and hyperbola, is always a third proportional to the transverse and conjugate axes; that is, $t : c :: c : p$, or

$$p=\frac{c^2}{t};$$

where p denotes the parameter, and t, c, the entire axes.

It may here, likewise, be observed, that besides the two axes above mentioned, any other diameter and its conjugate, either in the ellipse or hyperbola, are right lines drawn through the

Whence, if the second of these last equations be subtracted from the first, the result will give

$$4tz - 4t^2 = -4dx';$$

from which

$$\text{FE} = \frac{\text{AC}^2 - \text{FC} \times \text{CD}}{\text{AC}}, \text{ or}$$

$$z = \frac{t^2 - dx'}{t}.$$

7. Or, if FD, which is $=$ FC $-$ CD, be taken as the abscissa instead of DC, and put $= x''$, there will arise, by substituting $d - x''$ for x' in the last expression,

$$z = \frac{t^2 - d(d - x'')}{t} = \frac{t^2 - d^2 + dx''}{t}.$$

But, from what has been before shown, $t^2 - d^2 = c^2$, whence, also, we have

$$z = \frac{c^2 + dx''}{t}.$$

Which expression, together with that given above, furnishes two other equations, that may be considered as characteristic of the ellipse.

8. Also, if the angle EFD be denoted by ϕ, and the other parts of the triangle FDE, as before, we shall have

$$\text{rad.}(1) : \text{FE}(z) :: \cos.\text{EFD}(\phi) : \text{FD}(x''), \text{ or}$$

$$x'' = z \cos. \phi.$$

Whence, by substituting this value for x'' in the

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centre; and terminated on each side by the curve; the latter being so taken as to be parallel to the tangent at the vertex of the former.

Also that a right line FE, or fE, drawn from either of the foci to any point E in the curve, is called the *radius vector* of the curve; and the distance of the centre C from the focus F, or f, is called the *eccentricity* of the ellipse.

last expression, the result, when treated in the usual way, will give

$$z = \frac{c^2}{t - d \cos. \phi}.$$

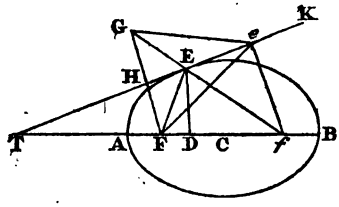
Which last equation, like all those that are derived from considering the co-ordinates as originating at the same point, or pole of the curve, is called the polar equation, and is of great use in the application of analysis to astronomy.

9. And if the angle EFD, or ϕ, in this case, be a right angle, its cosine will be $= 0$, and we shall then have

$$z = \frac{c^2}{t}.$$

Where t, c, being the semi axes, z, which now represents an ordinate at the focus, is evidently equal to half the parameter, agreeably to what has been observed in the preceding note.

10. Again, if it be required to draw a tangent to the ellipse, at any given point E, this may be done as follows:



From E, draw EF, Ef, to the two foci; and, in fE produced, take EG equal to EF, and join GF; then, if ET be drawn so as to bisect the angle GEF, it will be the tangent required.

For, in TE produced, take any other point e, and join eF, ef, and eG.

Then, since the two sides EG, EH, and the in-

cluded angle GEH, of the triangle GHE, are respectively equal to the two sides EF, EH, and the included angle FEH, of the triangle FHE, the side GH will be equal to HF, and the angle GHE to FHE (Euc. I, 4).

Also, because the two sides GH, H*e*, and the included angle GH*e*, of the triangle G*e*H, are respectively equal to the two sides FH, H*e*, and the included angle FH*e*, of the triangle F*e*H, the side *e*G will be equal to *e*F.

But, since, by Euc. I, 20, the sum of the sides *ef*, *e*G, or *ef* + *e*F, is greater than G*f*, or EF + E*f*, the point *e* must fall out of the curve, or otherwise, from the nature of the ellipse, *ef* + *e*F would be equal to EF + E*f*.

And as this is the case for every point whatever, that can be taken in the line TK, except the point E, it is evident that TK is a tangent to the curve at E (*o*).

11. From what has been here shown, it will also be easy to determine the point T, in which the tangent ET meets the axis BA produced.

For since the outward angle GEF, of the triangle

(*o*) Since the angle KE*f*, in this proposition, is equal to GET (Euc. I, 15), or TE*f*, it follows, that two lines drawn from the foci of an ellipse to the point of contact make equal angles with the tangent; and, consequently, as it is a well known principle in optics, that the angle of incidence is equal to the angle of reflection, it is evident, from what is above shown, that rays of light, issuing from one focus of an ellipse, and meeting the curve in every point, will be reflected into lines drawn from those points to the other focus; which is the reason why the points F, *f*, are called foci, or burning points.

FEf, is bisected by the right line ET, we shall have (Euc. VI, A,)

$$TF : Tf :: EF :: Ef.$$

Or, because $TF = CT - CF$, $Tf = CT + CF$, and by the description of the curve $Ef = AB - EF$, or $2CA - EF$, it will be as

$$CT - CF : CT + CF :: EF : 2CA - EF.$$

And, consequently, by taking the sum and difference of each antecedent and its consequent, we shall have

$$2TC : 2CF :: 2CA : 2CA - 2EF, \text{ or}$$

$$CT : CA :: CF : CA - EF.$$

But, from what has been before shown,

$$EF = \frac{CA^2 - CF \times CD}{CA};$$

which gives

$$CA(CA - EF) = CF \times CD, \text{ or}$$

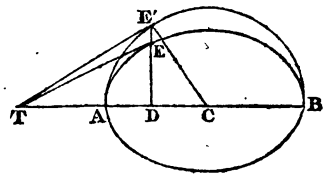
$$CA : CD :: CF : CA - EF.$$

Wherefore, by comparing this with the last proportion, there will arise

$$CT : CA :: CA : CD, \text{ or}$$

$$CD : CA :: CA : CT.$$

12. Hence, as CT is always a third proportional to CD and CA, it is evident, if the points D, A remain constant, that the tangent, drawn from the extremities E, E′, of the common ordinate DEE′, of any number of ellipses, described upon the same transverse axis AB, will all meet in the same point T.



And since among the series of ellipses, so formed, there must be one that has its two axes equal to each other, as in the circle, it follows that the above property will hold equally in this case as in the former; and thence furnishes the following elegant method of drawing a tangent to any point E in the curve of a given ellipse AEB.

From the centre C, with a radius equal to half the transverse axis AB, describe the semicircle AE′B, meeting the ordinate DE produced in E′; join the points C, E′, and draw E′T perpendicular to CE′, meeting BA produced in T; then, if the right line TE be drawn through the points T, E, it will be a tangent to the ellipse at E.

13. Also, if CA be now put $= t$, and CD $= x'$, as usual, we shall have, from what has been above shown,

$$\text{CD}(x') : \text{CA}(t) :: \text{CA}(t) : \text{CT, or}$$

$$\text{CT} = \frac{t^2}{x'}, \text{ and DT} = \frac{t^2 - x'^2}{x'}.$$

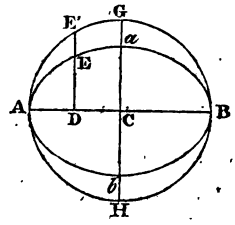
Which expressions give the values of CT, and the subtangent DT, both for the circle and ellipse, when the abscissa CD is reckoned from the centre C; and t, in the circle, is considered as the diameter.

14. Or, if AD be denoted by x, and $t - x$ be substituted for x' in each of the last equations, we shall have

$$\text{CT} = \frac{t^2}{t - x}, \text{ and DT} = \frac{2tx - x^2}{t - x}.$$

Which expressions are the values of CT, and the subtangent DT, for the circle and ellipse, when the abscissa AD is reckoned from the extremity of the transverse axis AB.

15. It may here, also, be farther shown, that the transverse diameter (AB) of any ellipse, is to the conjugate (ab), as the area of a circle, whose diameter is the transverse, is to the area of the ellipse.



For, by the general property of the ellipse before mentioned,

$$AC^2, \text{ or } CG^2, : Ca^2 :: AD \times DB : DE^2.$$

And, by the common property of the circle, (Euc. VI, 8, or VI, 13),

$$AD \times DB = DE'^2.$$

Whence, if this value be substituted for $AD \times DB$, in the last proportion, we shall have

$$CG^2 : Ca^2 :: DE'^2 : DE^2, \text{ or}$$
$$CG : Ca :: DE' : DE.$$

Which analogy will evidently hold for any ordinates whatever, that are drawn from different points in the transverse axis AB, parallel to DE.

And, consequently, as the circle and ellipse are composed of the same number of corresponding ordinates, which are all in the proportion of CG to Ca, or of AB to ab, it follows, from Euc. V, 12 and 17, that

$$AB : ab :: \text{circle on } AB : \text{ellipse } AEB.$$

But, by a well known rule, the area of the

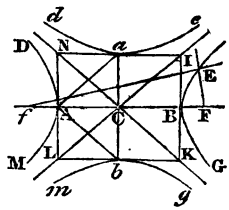
circle AE′B = .7854 × AB²; wherefore, if this value be substituted in the last proportion, we shall have

ellipse AEB = .7854 × *ab* × AB.

Which is the common expression usually given for its area.

OF THE HYPERBOLA.

1. LET AB be a given right line, and *f*, F, any two points, in its continuation, equally distant from A, B; then, if the two indefinite right lines *f*E, FE, cutting each other in E, be made to revolve in a given plane, about those points, so that their difference shall always be equal to AB, the locus of the point of intersection E will be an hyperbola EBG (*p*).



And if the two lines FE, *f*E, be turned in a contrary direction, and made to revolve, in a similar manner, about the same fixed points F, *f*, the locus of their point of intersection, in this case, will be

(*p*) The generation of this curve, like that of the ellipse, furnishes the following ready method of describing it mechanically by points.

Take any number of points in the prolongation of the transverse axis AB, and from the foci *f*, F, as centres, with the distances of A, B, from those points, describe as many arcs of circles as may be thought necessary; then, if a curve be drawn with a steady hand, through all the points of intersection of these arcs, it will be the hyperbola required.

another hyperbola DAM, equal to the former; or such, that if the parts of the plane were folded back upon each other, from the centre C, or middle of AB, the two curves would coincide.

Also, if an indefinite right line, *ab*, be drawn through the centre C, perpendicular to AB, and A*a*, A*b*, be each made equal to *f*C, or FC, the points *a*, *b*, will be the vertices of two other hyperbolas, *dae*, *mbg*, which may be traced in the same manner as the former, using *ab* instead of AB.

Where it is to be observed, that either of the opposite pairs of these curves, as EBG, DAM, are called opposite hyperbolas; and any two of them, which are adjacent to each other, as *dae*, EBG, are called conjugate hyperbolas.

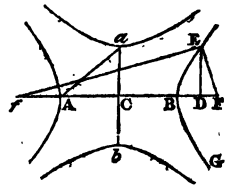
Also, the fixed points *f*, F, about which the revolving lines *f*E, FE, move, are called, as in the ellipse, the two foci; and the lines AB, *ab*, or *ab*, AB, are the transverse and conjugate diameters of the curves EBG and DAM, or of *dae* and *mbg*, according as they are referred to one or the other.

And, if tangents be drawn to the four vertices of the curves, or the extremities of the two axes, forming the inscribed rectangle NLKI, the diagonals NCK, LCI, of this rectangle, are called the asymptotes of the hyperbolas; being such lines, that, when produced, approach continually nearer to the curves, without ever meeting them, except they be considered as extended to an infinite distance.

2. It may here likewise be farther added, that when the transverse and conjugate axes AB, *ab*, are equal, in which case the asymptotes NK, LI, will

be perpendicular to each other, the curve is called an equilateral, or right angled hyperbola.

3. This being premised, let the semitransverse axis AC, or CB $= t$, the semiconjugate ac, or $cb = c$, fC, or CF $= d$, CD $= x'$, and DE $= y$.



Then, since fED, FED, are right angled triangles, we shall have, by Euc. I, 47,

$$fE^2 = fD^2 + DE^2 \text{ and } FE^2 = FD^2 + DE^2, \text{ or}$$
$$fE = \sqrt{\{(d + x')^2 + y^2\}} \text{ and } FE = \sqrt{\{(d - x')^2 + y^2\}}.$$

But, by the construction of the figure, $fE - FE =$ AB, or $2t$; whence

$$\sqrt{\{(d + x') + y^2\}} - \sqrt{\{(d - x')^2 + y^2\}} = 2t, \text{ or}$$
$$\sqrt{(d^2 + 2dx' + x'^2 + y^2)} = 2t + \sqrt{\{(d - x')^2 + y^2\}}.$$

Which last equation, by squaring each of its sides, and simplifying the result, will become

$$t\sqrt{(d^2 - 2dx' + x'^2 + y^2)} = dx' - t^2.$$

And, if this be again squared and reduced, we shall have

$$t^2y^2 = (d^2 - t^2)x'^2 - t^2(d^2 - t^2).$$

Whence, since fC, or A$a = d$, by construction, and $ac = c$, there will arise, by substituting c^2 for its equal $d^2 - t^2$,

$$t^2y^2 = c^2(x'^2 - t^2), \text{ or}$$
$$y^2 = \frac{c^2}{t^2}(x'^2 - t^2).$$

Which is the equation of the curve, when the

origin of the abscissa is taken at the centre of the hyperbola.

And, if BD be denoted by x, and $x+t$ be substituted for x', the last expression will then become

$$y^2 = \frac{c^2}{r^2}(2tx + x^2).$$

Which is the equation of the curve, when the abscissa BD is taken from the end of the transverse axis AB.

4. Hence, also, if this last expression be converted into a proportion, we shall obtain, by taking the whole diameters instead of their halves, the following proposition.

As the square of the transverse axis, is to the square of the conjugate, so is the rectangle of any two abscissæ to the square of their ordinate; that is,

$$AB^2 : ab^2 :: AD \times BD : DE^2.$$

Which theorem, together with that arising out of the description of the curve, may be considered, like those before given for the ellipse, as exhibiting the two fundamental properties of the hyperbola; or those from which most of the others can be derived.

And if the hyperbola be equilateral, in which case $AB = ab$, it is plain that the rectangle $AD \times BD$ will be equal to DE^2, or

$$y^2 = 2tx + x^2.$$

Which last equation differs from that of the circle only in its sign.

5. Again, if FE be put equal to z, $CD = x'$, and the distance CF, or cf, of the centre C from either

of the foci $=d$, as before, we shall have, by right angled triangles, and the description of the curve,

$$z=\sqrt{\{(d-x')^2+y^2\}}, \text{ and}$$
$$2t+z=\sqrt{\{(d+x')^2+y^2\}}.$$

And, consequently, by squaring each of these expressions, there will arise

$$z^2=d^2-2dx'+x'^2+y^2, \text{ and}$$
$$4t^2+4tz+z^2=d^2+2dx'+x'^2+y^2.$$

Whence, if the first of these equations be subtracted from the second, we shall have

$$4t^2+4tz=4dx', \text{ or}$$
$$z=\frac{dx'-t^2}{t}.$$

6. Or, if FD, which is $=$CF$-$CD, be taken as the abscissa, instead of CD, and put $=x''$, there will arise, by substituting $d-x''$ for x', in the last expression,

$$z=\frac{d^2-t^2-dx''}{t}.$$

But, from what has been before shown, $d^2-t^2=c^2$; whence, also, we have

$$z=\frac{c^2-dx''}{t}.$$

Which expression, together with that given above, furnishes two other equations, that may be considered as characteristic of the hyperbola.

7. Also, if the angle EFD be denoted by ϕ, and the other parts of the triangle as before, we shall have

rad. (1) : FE(z) :: cos. EFD(ϕ) : FD(x''), or

$$x''=z\cos.\phi.$$

Whence, by substituting this value for x'' in the

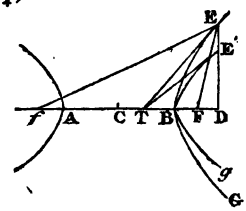
last expression, the result, when treated in the usual way, will give

$$z = \frac{c^2}{t + d \cos. \phi}.$$

Which is the polar equation of the hyperbola; being the same as that for the ellipse, except with regard to its signs; as is the case with all the rest.

8. And if FE, or z, in this expression, be considered as a perpendicular ordinate at the focus F, cos. ϕ will then become 0, and we shall have half the parameter $= \frac{c^2}{t}$, as in the ellipse.

9. It may likewise be farther shown, in a way perfectly similar to that before used for the ellipse, that a tangent ET, drawn from any point E in the curve, to meet the axis AB, will bisect the angle fEF, made by the right lines Ef, EF, drawn from E to the two foci (q).



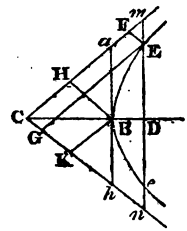
10. Also, if ED be an ordinate drawn from the point of contact E, perpendicular to AB produced, the semitransverse axis CB will be a mean proportional between CD and CT.

And as this proportion will hold for any other

(q) It may here be observed, as in the case of the ellipse, that if a polished surface were placed perpendicularly to the curve at the point E, a ray of light FE, or any perfectly elastic body, proceeding from one of the foci F, would be reflected by it in the direction Ef of the other focus.

hyperbola E′Bg, passing through the vertex B, it is plain that the tangents ET, E′T, &c. of any number of hyperbolas, having the same axis and abscissa, will all meet in the same point T.

11. To this we may add, that all the parallelograms GF, KH, &c. which are formed between the asymptotes c*m*, c*n*, and the hyperbola EB*e*, by lines FE, HB, GE, KB, drawn parallel to the asymptotes, are equal to each other.



For draw *ah* through the vertex of the curve, perpendicular to the semitransverse axis CB; and *a*B, or B*h*, as before shown, will be the semiconjugate; also draw *m*EDe*n* parallel to *ah*.

Then, by the equation of the curve, before given, when converted into an analogy, we shall have

$$CB^2 : Ba^2 :: CD^2 - CB^2 : DE^2.$$

And, because the triangles CB*a*, CD*m*, are similar, we shall also have

$$CB^2 : Ba^2 :: CD^2 : Dm^2,$$

or, by division,

$$CB^2 : Ba^2 :: CD^2 - CB^2 : Dm^2 - Ba^2.$$

Whence, since the first three terms of each of these proportionals are the same, DE^2 will be $= Dm^2 - Ba^2$, or

$$Ba^2 = Dm^2 - DE^2.$$

But DE being $= De$, and Dm, to Dn, $Dm^2 - DE^2$ will be $= (Dm + De) \times (Dn - De)$, or $me \times en$; wherefore, also,

$$Ba^2 = me \times en.$$

Again, because the triangles aHB, mFE, are similar, as, also, the triangles BKh, EGn, we shall have

$$HB : Ba :: FE : Em, \text{ or } en, \text{ and}$$
$$BK : Bh, \text{ or } Ba :: EG : En, \text{ or } me.$$

Whence, by multiplying the corresponding terms of these two proportions, there will arise

$$HB \times BK : Ba^2 :: FE \times EG : me \times en.$$

And, consequently, since it has been shown that Ba^2 is $= me \times en$, we shall have

$$FE \times EG = HB \times BK. \ldots\ldots (r)$$

But the parallelograms GF, KH, being equiangular, are to each other as the rectangles FE × EG and HB × BK; therefore,

parallelogram GF = parallelogram KH.

And the same may be shown of all the inscribed parallelograms, or rectangles, formed as above mentioned.

Cor. Hence it appears, that the asymptote Cm

(r) If the abscissæ and ordinates of an hyperbola, be referred to one of its asymptotes, instead of to the transverse axis, the property above mentioned furnishes a more simple equation of the curve, than either of those before given.

Thus, if HB, or CK, be put $= a$, KB $= b$, GE, or CF $= x$, and CG, or FE $= y$, we shall have

$$xy = ab, \text{ or } a : x :: y : b.$$

That is, the rectangle of the abscissa and ordinate is every where of the same magnitude; or any ordinate is reciprocally as its abscissa.

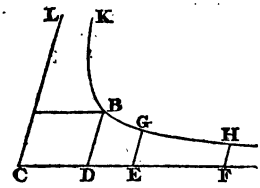
approaches continually towards the curve, without ever meeting it.

For since, whatever may be the length and breadth of the rectangle GF, we shall have

$$FE = \frac{HB \times BK}{EG},$$

it is plain, that, HB × BK being constant, FE must be always of some length, except when EG is infinitely great.

12. Also, if the abscissæ CD, CE, CF, &c. of any hyperbola KBGH, be taken on one of the asymptotes CF, in an increasing geometrical progression, the ordinates DB, EG, FH, &c. parallel to the other asymptote, will be in a decreasing geometrical progression, having the same ratio.



For, from what has been above shown, all the rectangles CD × DB, CE × EG, CF × FH, &c. being equal, the ordinates DB, EG, FH, will be reciprocally as the abscissæ CD, CE, CF, which are geometricals.

And the reciprocals of geometricals are, also, geometricals, having the same ratio; but decreasing in a converse order.

13. It may likewise be farther shown, that when the distances CE, CF, &c. taken on one of the asymptotes, as before, are in geometrical progression, the asymptotic spaces DEGB, DFHB, &c. will be in arithmetical progression; and are, therefore, analogous to the logarithms of the former.

Thus, supposing the hyperbola to be equilateral, or to have its asymptotes CF, CL, at right angles to each other, and that CD and DB are each $=1$, CE $=2$, CF $=4$, &c.; then will the area, or space, DEGB $=$ log. 2, and the area, or space, DFHB $=$ log. 4; and so on; the logarithms, in this case, being of the kind that are usually called *hyperbolic*, or *Neperian* logarithms, of which the modulus is unity; being equal to the area of the inscribed square CB.

And if the angle LCF, formed by the asymptotes, be considered as variable, any other system of logarithms may be derived in the same way.

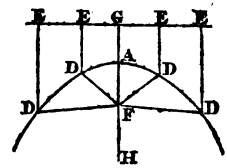
Thus, supposing the angle to be $25^\circ\ 44'\ 25''\frac{1}{2}$, and that CD and DB are each $=1$, CE $=2$, CF $=4$, &c. as before; the area of the inscribed rhombus CB, which is the modulus in this case, will be $=.4342944819$, and the asymptotic spaces DEGB, DFHB, &c. will be the common tabular logarithms of 2, 4, &c. (*s*)

OF THE PARABOLA.

1. If F be a given point, and EGE a right line given in position; then, if another point D be made to move in the same plane, so that the distance FD shall be always equal to the perpendicular DE,

(*s*) For a demonstration of the property of the curve abovementioned, which leads to the formation of these different systems of logarithms, see Traité Analytique des Sections Coniques, by the Marquis de l' Hospital, Book v, and the Elements of Plane Trigonometry, by Maseres, 1760, where this part of the subject is clearly and amply explained.

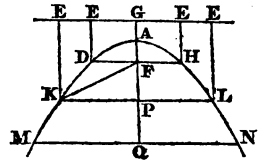
drawn from D to the given line, the locus of the point D will be a parabola (*t*).



Where it is to be observed, that the fixed point F is called the focus, and the right line EGE, which is at the same distance from the vertex A of the curve as the focus F, is called the directrix.

Also, the right line AH, passing through the vertex and focus of the curve, is called the axis of the parabola; and a double ordinate, drawn through the focus, is called the parameter.

2. Hence, let the parameter DH, of the parabola MAN, be put $=p$, the abscissa AP $=x$, and the ordinate PK $=y$; and join FK.



Then, since, by the description of the curve, $\frac{1}{2}p$, or FD = DE, or FG, and FG = 2AF, AF, or its equal AG, will be $=\frac{1}{4}p$.

(*t*) If the point F, in the annexed figure, be considered as the focus of an ellipse, or hyperbola; and the point D be made to move in the same plane, so that the distance FD shall always be to the perpendicular DE, drawn from that point to the directrix EGE, in a given ratio, the locus of the point D will be an ellipse when FD is less than DE, and an hyperbola when it is greater than DE.

Also, for a like reason, FK, which is = KE, or PG, will be $= x + \text{AF}$, or $x + \frac{1}{4}p$, and $\text{FP} = x - \text{AF}$, or $x - \frac{1}{4}p$.

But, by right angled triangles, $\text{PK}^2 = \text{FK}^2 - \text{FP}^2$, or

$$y^2 = (x + \tfrac{1}{4}p)^2 - (x + \tfrac{1}{4}p)^2.$$

Whence, by squaring each of the terms of the right hand side of this last expression, we shall have

$$\text{PK}^2 = (x^2 + \frac{1}{2}px + \frac{1}{16}p^2) - (x^2 - \frac{1}{2}px + \frac{1}{16}p^2), \text{ or}$$
$$y^2 = px.$$

Which is the equation of the parabola; where it appears that the parameter p is a third proportional to the abscissa and ordinate, as in the ellipse and hyperbola (*u*).

3. Also, if any other abscissa, as AQ, be put = X, and its corresponding ordinate MQ = Y, we shall have, by the above property,

$$\text{Y}^2 = p\text{X}, \text{ and } y^2 = px.$$

Wherefore, by comparing these two equations, there will arise

$$px : p\text{X} :: y^2 : \text{Y}^2, \text{ or}$$
$$x : \text{X} :: y^2 : \text{Y}^2.$$

(*u*) This theorem, like those for the ellipse and hyperbola, affords a ready method of describing the parabola by points.

Thus, since the distance of the extremity of any ordinate from the focus is equal to the distance of that ordinate from the opposite point G in the directrix, if a number of lines DH, KL, MN, &c. be drawn parallel to EGE, or perpendicular to the axis AQ, and the distances GF, GP, GQ, &c. be set off, each way, from the focus F to these lines, their points of meeting will be so many points in the curve.

Where it appears, that any two abscissæ are to each other as the squares of their ordinates.

Which theorem, as in the other curves before given, exhibits the leading property of the parabola.

4. In like manner, if AG, or AF, be put $=d$, FP $=x'$, FK $=z$, and the angle KFP $=\phi$, we shall have

$$z=2d\pm x'.$$

And, consequently, as in the former instances,

$$\text{rad.}(1) : \text{FK}(z) :: \cos.\text{KFP}(\phi) : \text{FP}(x'), \text{ or}$$

$$x'=z\cos.\phi.$$

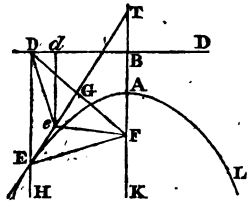
Whence, if this value be substituted for x' in the last expression, there will arise

$$z=2d\pm z\cos.\phi, \text{ or}$$

$$z=\frac{2d}{1\mp\cos.\phi}.$$

Which is the polar equation of the parabola; using the sign $+$, or $-$, according as the ordinate PK falls nearer to the vertex A than the focus, or farther from it.

5. Again, if it be required to draw a tangent to the parabola, at any given point E, this may be done as follows:



Join the point E and the focus F, and through E draw HED perpendicular to the directrix DBD, or parallel to the axis AK; then, if ET be drawn so as

to bisect the angle FED, it will be the tangent required (*x*).

For let any other point *e* be taken in ET, and having drawn *ed* perpendiculaar to DBD, join *e*D, *e*F, and DF.

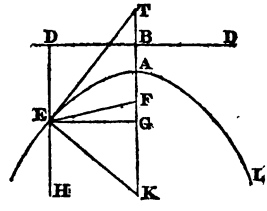
Then, since, by the description of the curve, ED is equal to EF, EG common to each of the triangles EDG, EFG, and the angle DEG equal to FEG, DG will be equal to GF, and the angle EGD to EGF.

Also, because the sides *e*G, GD, of the triangle EDG, are respectively equal to the sides *e*G, GF, of the triangle *e*FG, and the angles at G are equal, the side *e*D will be equal to *e*F.

But the angle *ed*D being a right angle, *e*D, or *e*F, will be greater than *ed*; whence the point *e* must fall out of the curve, or otherwise *e*F would be equal to *ed*.

And as this is the case for every point that can be taken in ET, except the point E, it is evident that ET will be a tangent to the curve at E.

6. From what has been above shown, it will also be easy to determine the point T, in which the tangent ET meets the axis produced.



(*x*) It may here be observed, that any right line EH, drawn within the curve, parallel to the axis AK, is called a diameter of the parabola; and the point E, where it cuts the curve, is called its vertex.

For draw HED parallel to AK, or perpendicular to the directrix DBD, as before; and make EG perpendicular to AK, and EK to ET.

Then, since ED is parallel to FT, the angle DET, or, as has been before shown, its equal TEF, will be equal to ETF; and, consequently, FT is equal to FE.

But, by the nature of the curve, FE is equal to ED, or GB; wherefore, also, FT is equal to GB.

Whence, if from these equals there be taken the equals AF, AB, there will remain AT equal to AG.

From which it appears that the subtangent GT, of any parabola EAL, is double the abscissa GA.

Also, because FT is equal to FE, and the angle KET is a right angle, the focus F will be the centre of a circle passing through the points T, E, K; and, consequently, FT, FE, and FK, will be all equal to each other.

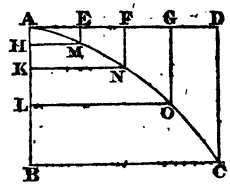
But since FE is equal to GB, or to GF + 2FA, and FK is equal to GF + GK, GF + 2FA will be equal to GF + GK, or GK equal to 2FA.

Hence it farther appears, that the subnormal GK is always the same constant quantity; being double the distance FA of the focus from the vertex, or half the parameter of the parabola.

So that, from what has been above shown, all parabolas, having the same vertex and abscissa, will have their tangents meet in the same point of the axis (*y*).

(*y*) Since KB, in the above figure, is perpendicular to the curve at E, or to the tangent ET, and the angle KBE is equal to KEE, or KEF, it is plain, from the optical principle before men-

7. To this we may add, that the area of any parabola, of the kind here treated of, is equal to two thirds of the area of its circumscribing parallelogram (z).



For let the semi-parabola ANC be circumscribed by the parallelogram BD; and suppose AD to be divided into an indefinite number of small equal parts AE, EF, FG, &c. and complete the parallelograms EH, FK, GL, &c.

Then, according to the method of indivisibles, the external space ANCDA of the figure, may be considered as composed of the sum of all the right lines EM, FN, GO, &c.

And since, by the property of the curve, AB : BC^2 :: AH, or EM, : HM^2, or AE^2; also, AB : BC^2

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tioned, that if the concave side of the parabola were a polished surface, all the rays of light, (as MN &c.) falling on that surface, parallel to the axis AK, (which, on account of his great distance, is the case with the rays of the sun,) would be reflected to the focus, or burning point, F.

(z) Besides the conic, or Apollonian parabola, here treated of, there are others of a higher kind, which are defined by the general equation

$$a^m x^n = y^{m+n};$$

where m and n may be any whole numbers whatever.

Thus, if $m=2$, and $n=1$, the expression, in this case, becomes $a^2x=y^3$, which is the equation of what is usually called the cubic parabola.

:: AK, or FN, : KN², or AF², &c. the several proportions, so formed, will give

$$EM = \frac{AB}{BC^2} \times AE^2$$

$$FN = \frac{AB}{BC^2} \times AF^2$$

&c. &c.

Wherefore, as $\frac{AB}{BC^2}$, in each of these cases, is the same constant quantity, we shall have the sum of all the lines

$$EM + FN \text{ \&c. } \ldots\ldots + DC,$$

or the space

$$ANCDA = \frac{AB}{BC^2}(O^2 + AE^2 + AF^2 \text{ \&c. } \ldots + DC^2),$$

reckoning the least term at A to be O, and the greatest to be DC.

Hence, since this last expression consists of a series of squares, whose roots are in arithmetical progression, we shall have, by putting $n =$ the number of its terms,

$$O^2 + AE^2 + AF^2 \text{ \&c. } \ldots\ldots + DC^2 = \frac{1}{3}n^3,$$

as is shown in Art. L, Vol. I, of the preceding part of the work; O being the first term, and n^2 the last.

But AD, which has been supposed to be divided into n equal parts, is $= n$, and, consequently, $\frac{1}{3}AD^3 = \frac{1}{3}n^3$; whence the space ANCDA, or

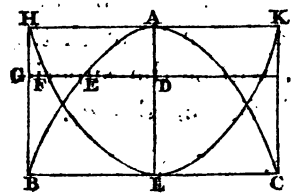
$$\frac{AB}{BC^2}(O^2 + AE^2 + AF^2 \text{ \&c. } \ldots + DC^2) = \frac{AB}{BC^2} \times \frac{AD^3}{3}.$$

Or, since BC is $=$ AD, the last expression, or the area of the external space ANCDA, will be $= \frac{1}{3}AB \times AD$;

and, consequently, the remaining space, or parallelogram

$$BD = \frac{2}{3} AB \times AD.$$

8. In the same manner it may also be shown, that the solid content of a paraboloid, or figure ABCA, generated by the rotation of a semi-parabola ABL, about its axis AL, is half the solidity of its circumscribing cylinder BK.



For put $p =$ the parameter; and, having described the equal paraboloid LKHL, draw GFED parallel to HA, or BL.

Then, by the equation of the parabola, before given, $p \times AD = DE^2$, and $p \times LD = DF^2$; whence, by addition,

$$p(AD + LD), \text{ or } p \times AL = DE^2 + DF^2.$$

But, by the same principle, $p \times AL = AH^2$, or DG^2; wherefore, also,

$$DG^2 = DE^2 + DF^2.$$

Whence, since circles are to each other as the squares of their radii, (Euc. XII, 2,) the circle described upon DG will be equal to the sum of the circles described upon DE and DF.

And the same property will hold for any section whatever, that is parallel to the base BL.

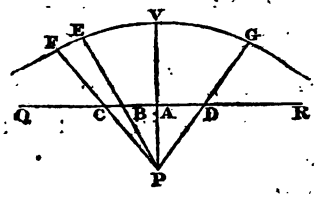
Hence, as the cylinder BA, and the two equal semiparaboloids BAL, HLA, are composed of the

sum of these sections, either of the latter will be equal to half the former, or the whole paraboloid

$$\text{BAC} = \tfrac{1}{2}\,\text{cylinder BK}.$$

OF THE CONCHOID.

Let PV, QR, be any two right lines, cutting each other at right angles in A; then, if PE, PF, PG, &c. be drawn from any fixed point P, in AP, so that AV, BE, CF, DG, &c. may be all equal to each other, the curve drawn through the several points F, E, V, G, &c. will be the superior conchoid, or the conchoid of Nichomedes (*a*).

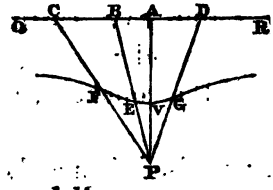


In like manner, if the equal lines AV, BE, CF, DG, &c. be taken on the other side of QR, so that VA shall be less than PA, the curve passing through

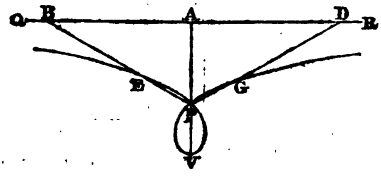
(*a*) Nichomedes, a Greek mathematician of the school of Alexandria, who was the inventor of the conchoid, also contrived an instrument for describing the curve by a continued motion, agreeably to the principles above laid down; which is highly praised by Newton in the Appendix to his Universal Arithmetic, who prefers it, for the construction of determinate equations of the third and fourth degree, (as is the case in finding two mean proportionals between two given right lines, and the dividing a given angle into three equal parts,) to any of the methods derived from the intersections of the conic sections.

He observes, "that certain curves are either not to be admitted into geometry, or else, in the construction of problems,

the points F, E, V, G, &c. in this case, is called the inferior conchoid.



And if the equal lines AV, BE, DG, &c. be still on the same side of QR, as in the last case, but VA greater than PA, the curve passing through the points E, P, V, G, &c. is called the nodated conchoid.



The fixed point P is called the pole, or centre,

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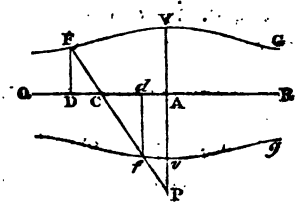
they are to be preferred to other lines of a more difficult description; for which reason (he adds) we approve of the trisections of an angle by a conchoid, which Archimedes, in his Lemmas, and Pappus, in his Collections, have preferred to all other inventions that have been employed for this purpose; because we ought either to exclude all lines, besides the circle and right line, out of geometry, or admit them according to the simplicity of their descriptions; in which case, the conchoid yields to none, except the circle."

Farther, "that is *arithmetically* more simple, which is determined by the more simple equations, but that is *geometrically* more simple, which is determined by the more simple drawing of lines; and, in geometry, that method ought to be reckoned the best which is geometrically the most easy: wherefore, I ought not to be blamed, if, with that prince of mathematicians, Archimedes, and other ancients, I make use of the conchoid in the construction of solid problems."

of the conchoid, and the right line QAR is called the directrix.

Also, the right line PAV, cutting the directrix perpendicularly in A, and meeting the curve in V, is called the axis of the conchoid, and V is the vertex of the axis.

1. This being premised, it may be readily shown that the directrix QR, of the conchoid FVG, or *fvg*, is an asymptote to the curve, or approaches continually nearer to it, without ever meeting it.



For, from any point F in the curve, draw FP to the pole P, and make FD perpendicular to the directrix QR.

Then, since the triangles CPA, CFD, are similar, we shall have PC : PA :: CF (or AV) : FD; whence

$$FD \times PC = PA \times AV.$$

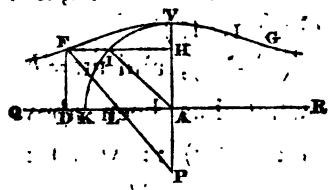
And, consequently, since the rectangle PA × AV is a constant quantity, the rectangle FD × PC will also be constant, whatever may be the values of FD and PC.

But it is evident, that PC increases as the distance of the point F from the axis PV increases; wherefore the distance FD of that point from the directrix must decrease in the same proportion, without ever becoming 0, or otherwise FD × PC would not be a constant quantity.

Hence the directrix QR, and the superior con-

choid FVG, will continually approach towards each other, without ever meeting; and the same method of reasoning may be applied to the inferior conchoid *fvg*.

2. It may also be further shown, that if any right line PF be drawn from the pole to the curve, and FH be drawn perpendicular to the axis PV, PF will be a fourth proportional to AH, AV, and PH.



For, since DH is a parallelogram, AH is equal to DF; and, by the description of the curve, AV is equal to LF; whence

AH : AV :: DF : LF.

But the triangles DFL, HPF, being equiangular, DF : LF :: PH : PF; and, consequently,

AH : AV :: PH : PF.

Which is the property of the curve that was to be shown.

3. Again, if with A as a centre, and AV as a radius, there be described the quadrant VIH, cutting FH in I, PH will be to HF as AH to HI.

For, since AI is equal to AV, being radii of the same circle, we shall have, from what has been above shown,

AH : AI :: PH : PF.

Hence, because the angle H, which is a right angle, is common to each of the triangles AHI, PHF, and the sides about the angles A and P are pro-

portional, the triangles will be equiangular, (Euc. VI, 7); and, consequently,

$$PH : HF :: AH : HI.$$

4. This being shown, the equation of the curve may be readily determined from the last proposition as follows:

Put $AV = a$, $PA = b$, DF, or $AH = x$, and $HF = y$.

Then, since, as before observed, AI is equal to AV, we shall have (Euc. I, 47)

$$HI = \sqrt{(a^2 - x^2)}, \text{ and } PH = b + x.$$

But, from what has been above shown,

$$b + x(PH) : HF(y) :: AH(x) : HI, \text{ or}$$

$$HI = \frac{xy}{b + x}.$$

Whence, equating this with the value of HI before found, we shall have, by multiplication and division,

$$y = \frac{(b + x)\sqrt{(a^2 - x^2)}}{x}.$$

Or, by clearing this last expression of surds, and simplifying the result, the equation of the superior conchoid FVG will, in that case, become

$$x^4 + 2bx^3 \pm (b^2 - a^2 + y^2)x^2 - 2a^2bx = a^2b^2.$$

And, by proceeding in a similar manner, the equations, both of the inferior and nodated conchoids, will be found to be

$$x^4 + 2bx^3 + (b^2 - a^2 + y^2)x^2 + 2a^2bx = a^2b^2.$$

The only difference between this and the former being the change of signs in the second and fourth terms; owing to $b + x$, in each of these cases, becoming $b - x$.

To which we may add, among other particulars

that might be mentioned, that, if $a = b$, the curve (fig. 3) will have a cusp in V, as the node between P and V will then vanish; and, in other cases, P will become a double point, from the curve passing twice through it.

Also, if PF, in the last figure, be put $= z$, the angle $P = \phi$, and the other letters as before, the polar equation will be readily found, from the above proposition, to be

$$z = \frac{b}{\cos. \phi} \pm a.$$

Where the upper sign is to be taken for the superior conchoid, and the under sign for the inferior and nodated conchoids.

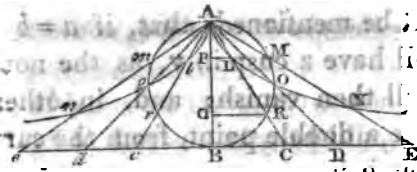
OF THE CISSOID.

At the extremity B of the diameter AB, of a given circle AOBO, erect the indefinite perpendicular EBE, and from the other extremity A draw any number of right lines AC, AD, AE, &c. cutting the circle in the points R, O, M, &c.; then, if CL be taken = AR, DO = AO, EN = AM, &c. the curve passing through the points A, L, O, N, &c. will be the cissoid of Diocles (b).

(b) Diocles, the inventor of this curve, was another Greek geometer of the school of Alexandria, but of a later date than Nichomedes.

He supposes the figure to be described by means of points, in a manner similar to that laid down in the third proposition next following; but Newton, in the Appendix to his Universal Arithmetic, has greatly improved the solution of Diocles, by showing how the curve may be described by a continued motion, as was done by Nichomedes for his conchoid.

It may here also be observed, that this curve, like that last mentioned, affords a ready method of finding two mean proportionals between two given right lines.



In which case, the circle AOBO is called the generating circle; and AB is called the axis of the curves ALON, &c. A*lon*, &c. which meet in a cusp at A, and, passing through the middle points O, *o*, of the two semicircles, tend continually towards the directrix *e*BE, which is their common asymptote.

1. Hence, if LP, RQ, be drawn perpendicular to the axis AB, it may be readily shown, from the description of the curve and the principles of common geometry, that

$$AP : PB :: PL^2 : AP^2,$$

For, by the construction of the figure, CL being equal to AB, QB will be equal to AP, and AQ to PB. And, consequently, since the triangles APL, AQR, are similar, we shall have

$$QR : AQ(PB) :: PL : AP, \text{ or}$$
$$QR^2 : PB^2 :: PL^2 : AP^2.$$

But, by the property of the circle, $QR^2 = AQ \times QB$, or $PB \times AP$; whence, by substitution,

$$PB \times AP : PB^2 :: PL^2 : AP^2, \text{ or}$$
$$AP : PB :: PL^2 : AP^2.$$

2. Whence, putting the axis $AB = a$, the abscissa $AP = x$, and the ordinate $PL = y$, we shall have

$$AP(x) : PB(a-x) :: PL^2(y^2) : AP^2(x^2), \text{ or}$$
$$x^3 = (a-x)y^2.$$

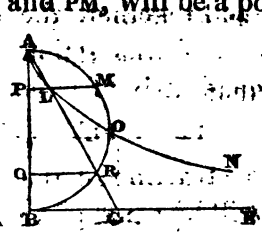
Which expression is the equation of the curve.

Also, since $a-x$ decreases as x increases, y, or y^2, must likewise continually increase, or, other-

wise, x^3, in the above equation, could not be equal to $(a-x)y^2$.

And because x can never be equal to a, as x^3, in that case, would be $=0$, instead of being $=a^3$, it is plain that y, or PL, will continually approach towards BE, without ever coinciding with it; whence BE is an asymptote to the curve.

3. If PM be drawn through any point L of the cissoid ALON, perpendicular to AB, and ALRC be drawn cutting the circle in R, the arc AM will be equal to BR; and, if AM be taken equal to BR, the intersection L, of AR and PM, will be a point in the curve.



For, by the construction of the figure before given, CL being equal to AR, or BQ to AP, PM must also be equal to QR; and, consequently, the arc AM equal to the arc BR.

Also, if the arc AM be taken equal to BR, BQ will be equal to AP, or BP to AQ; and, consequently, CL is equal to AR; which being the same as in the construction of the figure above mentioned, L will be a point in the curve.

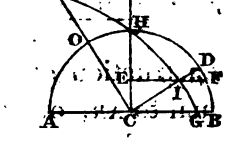
And by taking any number of equal arcs on the semicircle AOB, from A towards B, and from B towards A, as many points in the curve can be found as may be thought necessary.

COR. Since the arc AM is equal to BR, and O is the middle point of the semicircle, the arc OM will

be equal to OR; which arcs, therefore, may be used in the construction instead of the former (*b*).

OF THE QUADRATRIX.

Let AHB be a semicircle, through the centre C of which draw CM perpendicular to AB, and make HM equal to HC; then, if the radius CD, or CD produced, be supposed to move uniformly round C, from B towards H, while EF moves uniformly from CB towards M, continuing always parallel to itself, or perpendicular to CM, and so that the two motions begin and end together, the curve GIHL, described by the point of intersection I, of the lines CD, EF, will be the quadratrix of Dinostrates.



The circle AHB is called the generating circle, and the right line CM is called the axis of the quadratrix.

Also, if G be the point at which the lines CD, EF,

(*c*) Among other properties of the cissoid that might be mentioned, Newton, in his last Letter to Leibnitz, has shown how to find a right line equal to one of the legs of the curve, by means of the hyperbola; the investigation of which may be seen in his Fluxions.

He has also shown, that the infinitely long cissoidal space, contained between the asymptote *c*BE, and the curves NOLA, &c. ALO*n*, &c. in the first of the above figures, is equal to three times the area of the generating circle AOBO.

begin their motion, CG is called the base of the quadratrix; and MN, perpendicular to CM, is the directrix; which is an asymptote to the curve (*d*).

1. Hence, since the motions of the lines CD, EF, in the description of the figure above given, are uniform, and begin and end together, it is plain that any arc BD, passed over by CD, will be to the distance CE, passed over by EF, in the same time, as the quadrantal arc BH, is to the radius CH, or CE.

And, if CL be any other situation of the revolving line CD, after it has passed the point H, and KL be the corresponding situation of FE, the arc BHO will, also, be to the distance CK as the quadrantal arc BH is to the radius CH, or CE.

For the part of the curve HL, beyond the point H, being described by the same law as the part GH within the quadrant, it is evident that the arc BHO must have to the distance CK the same ratio as before.

And the same will be true of the remaining arc HB, and the distance HE, which are likewise to

(*d*) This curve, which was invented by Dinostrates, a disciple of the school of Plato, was called the quadratrix, on account of the property mentioned in one of the following articles, where it is shown, that the quadrantal arc, described from the centre C, with the base CG, is equal to the radius CB of the generating circle; so that, if it were possible to find the point G, by a simple geometrical operation, we should thence be enabled to determine the quadrature of the circle.

By another of its properties, also, a given angle, upon the supposition last mentioned, might be divided into any number of equal parts, or in any given ratio; it being shown in the

each other as the quadrantal arc HB is to the radius HC, or CB.

Or, since angles at the centre of a circle are as the arcs which measure them, it follows, by alternation, that HE is to HC as the angle HCD, or HCI, is to the angle HCB, or HCG.

Also, by equality, any abscissa HE is to any other abscissa, taken on the same line, as the angle HCD at the centre, corresponding with the former, is to the angle at the centre, corresponding with the latter; or as the arcs which subtend them.

2. From what is above shown, it is also farther evident, that, if the curve of the quadratrix GHL be produced, the revolving line CD will cut off greater and greater arcs from the generating circle BHA.

And, consequently, as the distances CE, CM, &c. of the moving line EF, must, in that case, continually increase, the distances EM, KM, &c. of the same line, from the directrix MN, will continually decrease.

Hence, the curve must approach continually nearer and nearer to MN; but can never meet it, for, otherwise, the revolving line would coincide with CA, and CA and MN would meet; which is impossible, since, by the construction of the figure, they are parallel to each other.

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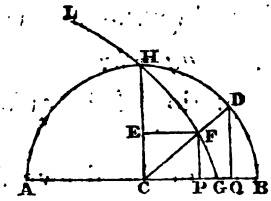
article next following, that CE, the distance passed over by the line EF, moving parallel to CB, is to the radius CH, or CB, as the arc BD is to the quadrant BH.

Tschirnhausen, a German mathematician, of the latter end of the seventeenth century, also formed a curve of this kind by means of points, which has similar properties to those above-mentioned.

Wherefore, the directrix MN is an asymptote to the quadratrix GHL, as was to be shown.

3. The base CG, of the quadratrix GHL, is a third proportional to the quadrantal arc BH of the generating circle AHB, and the radius CB.



For let the two moving lines CD, EF, cut each other in a point F of the curve, so that the arc BD of the generating circle may be indefinitely small; and draw DQ, FP, perpendicular to CB.

Then, by what has been before shown, the arc BD is to CE, as the quadrantal arc BH is to CB.

But since EF is a parallelogram, CE is equal to FP; and the arc BD, when taken indefinitely small, is equal to its sine DQ.

Whence, by substitution and similar triangles, DQ is to FP, or CQ to CP, as the quadrantal arc BH is to CB.

But when the arc BD is indefinitely small, and equal to DQ, CQ is equal to the radius CB, and CP becomes equal to CG.

Wherefore, by inverting the last analogy, the quadrantal arc BH is to CB as CB is to CG.

Cor. Since, CG : CB, or CH, :: CB, or CH, : quadrantal arc BH, and, by prop. I, CH : quadrantal arc BH :: CE : arc BD, we shall likewise have CG : CH :: CE : arc BD.

4. Hence, from what is above shown, we may readily obtain the equation of the curve, as follows:

Put the radius CH $= a$, the base CG, of the quadratrix, $= b$, the arc BD $= z$, and the ordinate PF, or CE, $= y$.

Then, from the last corollary, we shall have

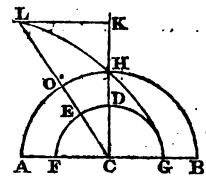
$$\text{CG}(b) : \text{CH}(a) :: \text{CE}(y) : \text{arc BD}(z);$$

whence, by multiplying the means and extremes,

$$ay = bz.$$

Which is the equation of the quadratrix, as was required.

5. If a right line CL be drawn from the centre of the generating circle, to any pont L in the quadratrix GHL, and the circle GDF be described with the base CG as a radius, the arc GE, of this circle, will be equal to CK, or that part of the axis which is contained between the centre C and the perpendicular LK.



For since, as before shown, CB : quadrantal arc BH :: CK : arc BO, and CG : CB :: CB : quadrantal arc BH, we shall have, by equality,

$$\text{CG} : \text{CB} :: \text{CK} : \text{arc BO}.$$

But, by the property of the circle, CG : CB :: quadrantal arc GD : quadrantal arc BH, or as arc GE : arc BO.

Hence, also, by equality, CK : arc BO :: arc GE : arc BO; and, consequently, arc GE = CK, as was to be shown.

Cor. From this it likewise farther appears, that

the quadrantal arc GD is equal to the radius CH, or CB, of the generating circle.

For since, as above, CG : CB :: CB : quadrantal arc BH, and CG : CB :: quadrantal arc GD : quadrantal arc BH, it follows, that CB : quadrantal arc BH :: quadrantal arc GD : quadrantal arc BH.

Whence, the second and fourth terms of this proportion being equal, the quadrantal arc GD will be equal to CB, or CH.

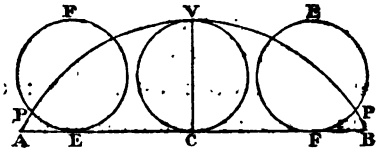
OF THE CYCLOID.

1. If a circle EPF, keeping always in the same plane, be made to roll along the right line AB, till a fixed point P, in its circumference, which, at first, touched the line at A, comes to touch it again, after a complete revolution, at B, the curve APVPB, described by the motion of the point P, is called a cycloid (e).

(e) The cycloid, or, as it was first called, the trochoid, is a curve of modern invention, the first knowledge of which, beyond that of its mere description, appears to be due to Galileo, who attempted, though without success, to determine its area; which, however, was afterwards done by Roberval and Torricelli; and the drawing of tangents to it was effected both by Descartes and Fermat.

To this we may add, that Pascal, in 1658, proposed, under the feigned name of Detonville, several problems relating to this curve, to the mathematicians of that time, offering a prize to any one who should answer them within a given time; which was claimed by Wallis, but refused on account of some mistake in his calculation.

Other solutions, without any regard to the prize, were like-



The circle EPF is called the generating circle; and the right line AB, on which it revolves, is called the base of the cycloid.

And as every part of the circumference of the circle touches this line in succession, it is evident that the base AB is equal to the circumference of the generating circle EPF.

Also, the right line, or diameter, CV, of the circle, which bisects the base AB at right angles, is called the axis of the cycloid; and the point V, where it meets the curve, is the vertex of the cycloid.

2. If P be a point in the fixed diameter AF produced, and the circle AEF be made to roll along the line AB as before, so that the point A, which first touches it at one extremity, shall touch it again at B, the curve PVP, described by the point P, is called the *curtate* cycloid.

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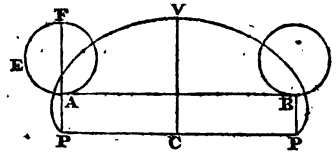
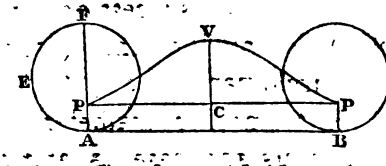
wise given by Sluse, Huygens, and sir Christopher Wren; the latter of whom appears to have been the first who discovered the absolute rectification of an arc of the cycloid. It may also be observed, that Wallis, in his treatise upon the cycloid, and Huygens in his *Horologium Oscillatorium*, have treated fully of the nature and properties of this curve; and John Bernouilli found it to be the curve of swiftest descent. — See Montucla, *Histoire des Mathématiques*, new Edition, Tom. II, p. 52, et seq. where the reader will find a full detail of all the most curious particulars relating to the history of this celebrated curve.

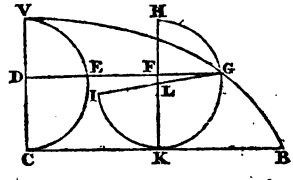


3. And if the point P be any where in the unproduced diameter AF, and the circle AEF be made to roll along AB, from A to B, as above, the curve PVP, in this case, is called the *prolate*, or *inflected* cycloid.



Where it is to be observed, that, in the two last figures, the generating circle, the base, and axis, are denoted by the same letters as in the common cycloid.

4. If an ordinate GD, be drawn from any point G, in the cycloid VGB, perpendicular to the axis VC, that part of it, EG, which lies between the circle and the curve, will be equal to the arc VE.



For let HGKI be the position of the generating circle, when the describing point is at G, and the point of contact at K; and through the centre L draw the diameters HK, GI.

Then, since HK is perpendicular to CB, and GD to VC, CF will be a parallelogram; and, consequently, CD is equal to KF, and DF to CK.

But, from what has been before observed, the semicircle GKI is equal to the semibase CB; and, by the description of the curve, the arc GK is equal to BK.

Hence, also, the remaining part of the base CK must be equal to the remaining part of the arc KI, or to its equal HG.

Also, since CD is equal to KF, and, consequently, DV to FH, it is evident, from the nature of the circle, that DE is equal to FG, and the arc VE to HG.

Wherefore, since CK, or its equal DF, or EG, has been shown to be equal to the arc HG, EG will also be equal to the arc HG, or VE, as was to be shown.

Cor. Any ordinate DG, of the cycloid VGB, is equal to the sum of the circular arc VE and its sine ED.

5. Hence, from what is above shown, the equation of the cycloid may be immediately obtained, as follows:

Put the ordinate DG $= y$, the arc VE $= z$, and its sine ED $= s$.

Then, by the last corollary, we shall have

$$y = z + s.$$

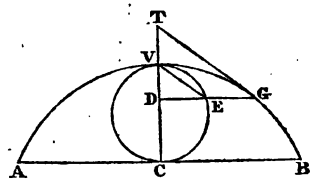
Which is the equation of the common cycloid VGB.

And if, in the cartate or prolate cycloid, a be put = circumference of the circle described upon the axis VC, $b =$ base AB, $y =$ any ordinate, $z =$ circular arc cut off by it, and $s =$ its sine, it may be shown, by a mode of reasoning similar to that before used, that

$$y = \frac{bz}{a} + s.$$

Which is the equation of either of the two cycloids last mentioned (*f*).

6. To this we may also add the following curious properties of this elegant and useful curve; which, as they cannot, except the first, be readily derived from the common principles of algebra, are here given without demonstration.



1. If DG be any ordinate, cutting the cycloid in G, and the circumference of the generating circle in E, the circular arc VE will be equal to the right line EG, as has been above shown; and the semicircumference VEC is equal to the semibase CB.

2. The right line GT, drawn from any point G in the curve, parallel to the chord EV, is a tangent to the cycloid at that point.

3. The cycloidal arc VG is double the chord VE; and the semicycloidal arc VB is double the diameter VC.

4. The area of the cycloid AVBA is triple the

(*f*) In addition to what is here said, it may be farther observed, that if the circumference of one circle be made to roll along the circumference of another, either on the outside or the inside, the curve described by any given point in its circumference is called an *epicycloid;* which is a curve of great use in mechanics, being that commonly used in forming the teeth of wheels, and for other purposes; but as this is of the transcendental kind, its properties do not admit of a simple algebraical investigation.

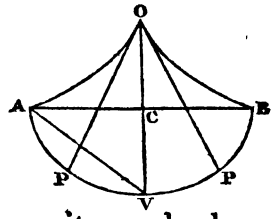
area of the generating circle VECV; and, consequently, the three spaces AVCA, the circle VECV, and VBCV, are all equal to each other.

5. The upper segment of a cycloid, cut off by a right line parallel to the base, at the distance of $\frac{1}{4}$ of the axis from the vertex, is equal to the regular hexagon inscribed in the generating circle.

6. The solid, generated by the revolution of the cycloid about its base AB, is to its circumscribing cylinder as 5 to 8.

7. The centre of gravity of the whole cycloid AVB is in the axis VC, at the distance of $\frac{5}{8}$ of that line from the vertex V.

8. If AVB be any reversed cycloid, and OA, OB, two other equal semicycloids, meeting at the point O, and each half the length of OA or OB, a pendulum OV, of twice the length of the axis CV, being suspended at O, and made to move backwards and forwards, through AVB, by the string plying round the two curves, will perform all its vibrations, whether the arcs it describes be greater or smaller, in the same time.



9. In this case, it can also be proved, that the time of one vibration, in any arc of a cycloid, is to the time of a body falling perpendicularly through the axis CV, or half the length of the pendulum, as the circumference of a circle is to its diameter, or as 3.14159 to 1.

So that if π be put $=3.14159$, $l=$ the length of the pendulum, and $g=$ space fallen through by a heavy body in one second of time, we shall have, by the well known law of gravity, in this case,

$$\sqrt{g} : \sqrt{\tfrac{1}{2}l} :: 1'' : \sqrt{\frac{l}{2g}}.$$

The last term of which proportion is equal to the time of falling through $\frac{1}{2}l$.

And, consequently, putting $t=$ time of one vibration, the analogy before mentioned will give

$$1 : \pi :: \sqrt{\frac{l}{2g}} : t, \text{ or}$$

$$t = \pi\sqrt{\frac{l}{2g}}. \quad . \quad . \quad . \quad . \quad (g)$$

From which, π and $2g$ being constant, it farther

(g) It may be here worth observing, that, if the length of the second's pendulum be first ascertained by trial, which is the most easily done, this theorem will give the descent of a body, by the force of gravity, in a second, more accurately than it can be obtained by direct experiment.

Thus, the length of a pendulum that vibrates seconds, in the latitude of London, has been found to be $39\frac{1}{8}$ inches; whence, if this be written for l in the above theorem, we shall have

$$\pi\sqrt{\frac{39\frac{1}{8}}{2g}} = 1''.$$

Which gives, by the common method of resolving an equation of this kind,

$$g = \tfrac{1}{2}\pi^2 l = \tfrac{1}{2}\pi^2 \times 39\tfrac{1}{8} = 193.07 \text{ inches.}$$

Or $16\frac{1}{12}$ feet, for the descent of a body by gravity in a second.

And if, on the contrary, g be known, and the length of the pendulum be required, we shall have

$$l = \frac{2g}{\pi^2} = \frac{386.14}{\pi^2} = 39.126 \text{ inches}$$

for the length of the pendulum vibrating seconds in the latitude of London.

appears, that the times of vibrations of pendulums, in cycloidal arcs, are as the square roots of their lengths, whatever may be their weights; or

$$t : t' :: \sqrt{l} : \sqrt{l'}.$$

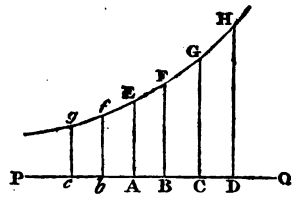
And if the pendulum vibrate in small arcs of a circle, which it is evident will nearly coincide with the small cycloidal arcs at the vertex V, the times, in this case, will also be nearly isochronus; and, consequently, the above formulæ are equally applicable to the circle and the cycloid, as far as regards any practical purposes.

10. Lastly, the cycloid has the remarkable property, mentioned in the last Note, of being the curve of swiftest descent: that is, a body will descend through the arc of the curve APV, from A to V, in less time than it would pass over the right line AV, or by any other rout.

OF THE LOGARITHMIC CURVE.

If from any fixed point A, in the indefinite right line PQ, there be taken a number of parts AB, AC, AD, &c. in arithmetical progression, and the perpendicular ordinates AE, BF, CG, &c. drawn from the points A, B, C, &c. be taken in geometrical progression, the line EFGH, &c. which passes through their extremities, is called the logarithmic curve (*h*).

(*h*) This curve, of which Gunter is commonly said to have given the first idea, greatly facilitates the conception of logarithms to the imagination, and affords an obvious proof of the very important property, that the fluxion of any number,



1. Hence, agreeably to the description of the figure, here laid down, the line PQ of the abscissæ, which is called the axis, may be shown to be an asymptote to the curve, as follows:

Let the abscissæ A*b*, AC, &c. taken on the left of AE, make a part of the same arithmetical progression with AB, AC, AD, &c. on the right; and the ordinates *bf*, *cg*, &c. a part of the same geometrical progression with AE, BF, CG, &c. but so as to decrease while the others increase.

Then, because the square of any term of a continued geometrical progression is equal to the rectangle of any two terms that are equally distant from it (Art M, Vol. II), we shall have

$$bf \times \text{BF}, \text{ or } cg \times \text{CG}, \text{ \&c.} = \text{AE}^2.$$

And, consequently, since the square of AE is a constant quantity, any one of the rectangles $bf \times$ BF, $cg \times$ CG, &c. will also be constant, whatever may

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or quantity, is to the fluxion of its logarithm as the number itself is to the subtangent.

It may be farther observed, that Huygens, in his *Dissertatio de Causa Gravitatis*, as well as several later writers, have treated very fully of all the principal properties of this curve, and shown their analogy to logarithms; but, as has been observed in the Introduction to my Treatise on Plane and Spherical Trigonometry, the doctrine of logarithms, though it may admit of a clear illustration in this way, has no necessary connexion with this or any other geometrical figure.

be the values of the ordinates of which they are composed.

But, by the description of the figure, the ordinates bf, cg, &c. decrease as BF, CG, &c. increase; wherefore, the right line PQ and the curve gEH must continually approach towards each other, without ever meeting; or otherwise the rectangles above mentioned could not be always equal to AE^2.

Whence, the axis PQ is an asymptote to the curve, as was to be shown.

Cor. As the abscissæ AB, AC, AD, &c. in the above figure, constitute a series of quantities in arithmetical progression, and their ordinates AE, BF, CG, &c. a corresponding series in geometrical progression, the former may be considered as a set of natural numbers that are analogous to the logarithms of the latter; from which property the curve has received its name.

2. The equation of the curve may, also, be readily derived from the above description of it in the following manner:

Put AE $= 1$, and BF $= a$; then, from the nature of the figure, we shall have

$$1 : a :: a : \text{CG},\ a : a^2 :: a^2 : \text{DH}, \text{ \&c.}; \text{ or}$$
$$\text{CG} = a^2,\ \text{DH} = a^3,\ \text{\&c.}$$

Whence, if x be made to denote any number of equal parts AB, BC, CD, &c. of the axis PQ, a^x will evidently be equal to the ordinate drawn from the extremity of the segment which is represented by x.

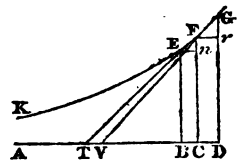
And, consequently, if this ordinate be put equal to y, we shall have

$$a^x = y$$

for the equation required; from which, on account

of its form, the figure is sometimes called the *exponential curve*.

3. The subtangent of the logarithmic curve is always of the same length, from whatever point of the curve the tangent may be drawn.



For let ET, FV, be any two tangents to the curve KEFG, at the points E, F, which are indefinitely near each other; and having drawn the equidistant ordinates EB, FC, GD, make E*n*, F*r*, parallel to AD.

Then, since BC = CD, we shall have, from the description of the curve, BE : CF :: CF : DG; or, by division, BE : CF :: CF − BE : DG − CF (Euc. v, 19).

But CF − BE = *n*F, and DG − CF = *r*G; wherefore BE : CF :: *n*F : *r*G; or, alternately, BE : *n*F :: CF : *r*G.

Also, because the triangle F*n*E is similar to EBT, and the triangle G*r*F to FCV, BT : *n*E :: BE : *n*F, and CV : *r*F :: CF : *r*G.

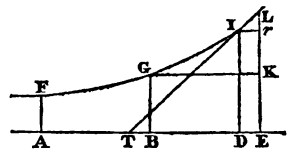
Hence, since BE : *n*F :: CF : *r*G, as before shown, we shall also have BT : *n*E :: CV : *r*F (Euc. v, 11).

But CB being = CD, *n*E will be = *r*F; and, consequently, the other two terms of the proportion, or the subtangents BT, CV, will be equal.

And the same will be true for any other points in the curve.

Note. The constant subtangent of this curve is what was first called, by Cotes, the modulus of the system of logarithms.

4. The logarithmic space BELG, comprehended between any two ordinates BG, EL, is equal to the rectangle of the subtangent TE, and the difference LK, of those ordinates.



For let the ordinate DI be indefinitely near EL, and draw I*r* parallel to AE.

Then, if LT be a tangent to the curve at L, the triangles LTE, LI*r*, will be similar.

Whence, since EL : ET :: *r*L : *r*I, we shall have EL × *r*I = ET × *r*L.

But DI being indefinitely near EL, the rectangle EL × ED, or EL × *r*I, = the area of the space DELI; wherefore, also, ET × *r*L = DELI.

Hence, since, as before shown, ET is a constant quantity, the sum of all the spaces DELI, or the whole area BELG is = ET × sum of all the L*r*'s, or = ET × LK.

Cor. From this it appears, that the area of the whole logarithmic space, indefinite towards A, is double the triangle LET.

5. To this we may add, that the solid formed by the revolution of the infinitely long space FLEA about the axis AE, is equal to half a cylinder of the base LE, and of the altitude of the constant subtangent ET.

6. Also, the logarithms of equal numbers are proportional to the subtangents of the curves to which they belong; the latter being the moduli of the different systems of logarithms derived from those curves.

7. And if any large numbers, as 10000000, 10000001, 10000002, &c. differ but little from each other in value, their differences will be nearly proportional to the differences of their logarithms.

OF THE SPIRAL OF ARCHIMEDES.

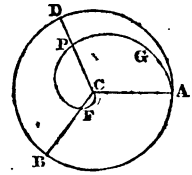
If any right line CA be moved uniformly, in the same plane, round the centre C, describing the circle ABDA; and, at the same time, a point P be supposed to move uniformly along that line, from C towards A, so that both motions begin and end together, the curve CFPGA, described by that point, is called the spiral of Archimedes.

And if the same line produced, be made to move round a second time, while the describing point P continues its motion as before, the curve described, in this case, will be the second spiral; and so on (*i*).

(*i*) The first treatise on a spiral, or figure of the kind above mentioned, was given by Archimedes; in which he determined the relation of its area to that of the circumscribed circle, as also the relations of its sectors, and other particulars.

After him, Pappus, in his Mathematical Collections, considered the nature of a spiral which is described, by similar motions, on the surface of a sphere, and found a portion of the surface, terminated by this spiral, to be equal to the square of the diameter of the sphere.—See Maclaurin's Fluxions, Introduction, p. 31—33.

Dr. Halley has, also, treated of such spirals as are formed



The circle ABDA is called the generating circle, and the point C is called the centre of any one of the spirals, described as above mentioned.

Also, a right line, drawn from C to the point where each of the spirals end, is called the axis of that spiral; and a right line, drawn from C to any other point in the spiral, is called an ordinate.

1. If F be any point in the curve, and CFB be drawn, cutting the circle in B, then will CF : radius CB, or CA : : arc AB : circumference ABDA.

For, by the description of the curve, the right line CA, in which the fixed point A is situated, revolves about C with an uniform motion; and the point which describes the spiral also moves from C towards A with an uniform motion.

Whence, the circumference of the circle ABDA, and the axis CA, are described by uniform motions in the same time.

And, for similar reasons, the arc AB, and the ordinate CF, are described by the same uniform velocities in the same time.

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by the Rhumb lines on the surface of the globe; which, on account of their making equal angles with every meridian, will also make equal angles with the meridians in the stereographic projection, formed on the plane of the equator; and must therefore, as he observes, be proportional spirals about the polar point: from which he has demonstrated, that the meridional line is a scale of logarithmic tangents of the half complements of the latitude.— See next Article.

But spaces that are described with the same uniform velocities are to each other as the times in which they are described.

Wherefore the circumference of the circle ABDA : arc AB :: time of a whole revolution : time of describing the arc AB.

Hence, also, since CA, or CB, is to CF as the above mentioned times are to each other, the circumference ABDA : arc AB :: CB, or CA : CF; or, inversely, CF : CB, or CA :: arc AB : circumference ABDA.

Cor. If any other ordinate CP be drawn, and produced till it cuts the generating circle in D, the arc AB : the arc ABD :: CF : CP.

From which it is farther obvious, that if any number of arcs, or angles about C, be taken in arithmetical progression, the ordinates, or distances of the moving point from C, will, also, be in arithmetical progression.

2. Hence, if the axis of the spiral, or, which is the same thing, the radius CA of the generating circle, be put $=r$, the circumference of that circle $=\pi$, any arc AB $=z$, and the corresponding ordinate CF of the curve $=y$, we shall have

$$y : r :: z : \pi, \text{ or}$$

$$y = \frac{rz}{\pi}.$$

Which is the equation of the first spiral CFPGA.

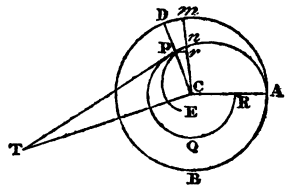
Also, retaining the same letters as before, it may readily be shown, that the equation of the second spiral is

$$\pi y = r(\pi + z).$$

And if n be made to denote the number of spirals, the equation of any one of them will be

$$\pi y = r\{\pi(n-1) + z\}.$$

3. If the right line CPD be drawn through any point P in the spiral EPA, and CT be made perpendicular to it, and equal to a fourth proportional to the radius CD, the arc ABD and the ordinate CP, the right line PT, joining the points P, T, will be a tangent to the curve at P.



For draw Cm indefinitely near CD, cutting the spiral in n, and make Pr perpendicular to Cm.

Then, by the corollary to the last proposition, the arc ABD : CP :: arc ABDm : Cn; or, by division, arc ABD : CP :: arc Dm : nr; whence

$$nr = \frac{\text{CP} \times \text{arc D}m}{\text{arc ABD}}.$$

And since, by similar sectors, CD : CP :: Dm : Pr, we shall also have

$$\text{P}r = \frac{\text{CP} \times \text{arc D}m}{\text{rad. CD}}.$$

But if TP touches the curve at P, the triangles nPr and TPC will be similar; wherefore

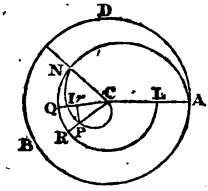
$$nr\left(\text{or } \frac{\text{CP} \times \text{arc D}m}{\text{arc ABD}}\right) : \text{P}r\left(\text{or } \frac{\text{CP} \times \text{arc D}m}{\text{rad. CD}}\right) :: \text{CP} : \text{CT};$$

$$\text{or rad. CD} : \text{arc ABD} :: \text{CP} : \text{CT}.$$

Cor. If the arc PQR be described with the radius CP, and the perpendicular CT be made equal to it, TP will be a tangent at P.

For, by similar sectors, CD, or CA : CP :: arc DBA : arc PQR; and, from what has been before shown, CA : arc DBA :: CP : CT; wherefore CT = arc PQR.

4. The spiral space CNPC is one third of the area of the sector CNRL, whose radius is CN and arc NRL.



For draw CIQ indefinitely near CPR, and make Pr perpendicular to CQ.

Then, since CR^2, or $CN^2 : CP^2 ::$ sector CRQ : spiral CPI, we shall have

$$\text{area of the spiral CPI} = \frac{CP^2 \times \text{sector CRQ}}{CN^2}.$$

Therefore, the sum of all the sectors CPI, will be equal to the sum of all the expressions

$$\frac{CP^2 \times \text{sector CRQ}}{CN^2}.$$

But if the arc NRL be divided into an indefinite number of equal parts, each equal to QR, and right lines be drawn to them from the centre C, all the sectors formed by them will be equal; and all the CP's will be in arithmetical progression.

Wherefore, by Art. P, vol. II, the sum of all the CP^2's $= \frac{1}{3} CN^2 \times$ by the number of them; and the sum of all the $\frac{CP^2\text{'s} \times \text{sector CRQ}}{CN^2} = \frac{1}{3}$ of all the sectors CRQ multiplied into the number of them.

Whence the sum of all the CPI's, or the spiral space CNPC $= \frac{1}{3}$ of all the sectors CRQ multiplied by the number of them, or $= \frac{1}{3}$ whole sector CNRL.

COR. The whole spiral space CPNA is equal to $\frac{1}{3}$

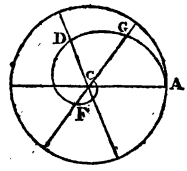
of the generating circle ABDA; since CN, in this case, becomes equal to the radius CA.

It may, also, be farther shown, that if the area of the generating circle be A, the first, second, third, fourth, &c. spiral spaces will be

$$\frac{1}{3}A,\ \frac{7}{3}A,\ \frac{19}{3}A,\ \frac{37}{3}A,\ \&c.$$

OF THE LOGARITHMIC SPIRAL.

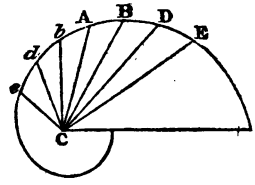
If the curve CFDGA be drawn about the fixed point C, so that it may every where cut the right lines, or radii, CF, CD, CG, &c. in the same angle, the curve, so formed, is called the logarithmic, or proportional spiral (*k*).



Where it may be observed, that the point C is called the centre of the spiral, and the right lines CF, CD, CG, &c. drawn from the centre to the curve, are called the ordinates.

(*k*) Descartes, according to the testimony of Montucla, appears to have been the first inventor of this elegant curve, an account of which he sent, in a letter to Mersenne, in 1638, who, agreeably to his custom, upon similar occasions, communicated it to some of the most eminent geometers of that time, several of whom investigated its principal properties, and even discovered some that were unknown to the proposer, particularly that the curve, though it makes an infinite number of revolutions round its centre, without ever arriving at that point, is yet of a finite length; being equal to a certain right line, as is shown in the third proposition next following.

1. If any number of ordinates CA, CB, CD, CE, &c. be drawn to the logarithmic spiral ABDE &c. so as to make equal angles at the centre C, they will be in continued geometrical progression.



For if the equal angles ACB, BCD, DCE, &c. be taken indefinitely small, the portions of the curve AB, BD, DE, &c. may be considered as right lines.

And, therefore, since the angles CAB, CBD, CDE, &c. are supposed to be equal, by construction, the triangles CBA, CDB, CED, &c. will be similar; or

CA : CB :: CB : CD, and CB : CD :: CE, &c.

Also, if the ordinates *cb*, *cd*, *ce*, &c. on the left of CA, be made to decrease, while those on the right increase, and the angles ACB, A*cb*, *bcd*, *dce*, &c. be all equal to each other, it may be shown, as above, that

CB : CA :: CA : C*b*, and CA : C*b* :: C*b* : C*d*, &c.

Cor. Hence it also follows, as in the logarithmic curve, that CA being a mean proportional between any two ordinates that are equally distant from it, the spiral must make an infinite number of revolutions before it can come to the centre C.

2. Also, since the angles ACB, ACD, ACE, &c. are in arithmetical progression, and the ordinates CA, CB, CD, CE, &c. are in geometrical progression, the former are analogous to the logarithms of the latter; from which circumstance the curve receives its name.

From this property it is likewise farther evident, that, in any system of logarithms, it will be as

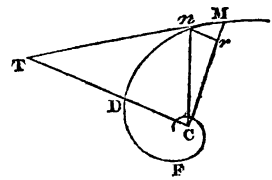
$$\angle \text{ACB} : \angle \text{ACD} :: \log.\frac{\text{CB}}{\text{CA}} : \log.\frac{\text{CD}}{\text{CA}}.$$

And from what is above shown, if CA be put $=1$, CB $=a$, $x=$ any angle in the arithmetical series, and $y=$ the corresponding ordinate, we shall have

$$a^x = y.$$

Which is the equation of the spiral; being the same as that for the logarithmic curve.

3. If MT be a tangent to the curve at M, and CT be drawn perpendicular to the ordinate CM, the length of the whole spiral MDFC will be equal to the tangent MT.



For draw Cn indefinitely near CM, and make nr perpendicular to CM.

Then, since, in the right angled triangle Mnr, all the angles are given, Mr will be to Mn in a given ratio, which let be that of r to s; and, in this case, we shall have

$$\text{M}n = \frac{s}{r}\text{M}r, \text{ and all the M}n\text{'s} = \frac{s}{r} \times \text{ all the M}r\text{'s}.$$

That is, the whole length of the curve

$$\text{MDFC} = \frac{s}{r} \times \text{MC}.$$

But the triangles Mnr and MTC being similar, MT : MC :: Mn : Mr, or as $s : r$; whence

$$\text{MT} = \frac{s}{r} \times \text{MC} = \text{whole length of the spiral MDFC}.$$

COR. If the angle CMT be taken = the angle which the spiral makes with any one of the ordinates CD, and CT be drawn perpendicular to CM, TM will be a tangent to the curve at M.

For the tangent TM, and the indefinitely small part of the curve at M, may, in this case, be supposed to coincide.

From which it also follows, that TM is to CM every where in the same given ratio.

4. Again, if MT be a tangent at the point M, and CT be perpendicular to the ordinate CM, the area of the spiral MDFC, described by a radius making an infinite number of revolutions, will be equal to half the triangle TMC.

For draw the ordinate Cn indefinitely near CM, and make nr perpendicular to CM, as before.

Then, since all the angles of the triangle Mnr are given, Mr will be to nr in a given ratio, which let be that of r to s; and we shall have

$$nr = \frac{s}{r} \times \text{M}r, \text{ and } \triangle \text{MC}n = \frac{nr \times \text{MC}}{2} = \frac{s}{2r} \times \text{M}r \times \text{MC}.$$

Hence, if MC be divided into an indefinite number of equal parts, each equal to Mr, the sum of all the Mr's, to C, will be equal to as many times the number of them as is denoted by MC; that is, the sum of the Mr's, to C = MC.

Wherefore the sum of all the $\frac{s}{2r} \times \text{M}r \times \text{MC} = \frac{s}{2r} \times \frac{\text{MC}^2}{2}$, or $\frac{s}{4r} \times \text{MC}^2$; and, consequently, the sum of all the triangles MCn, or the whole area of the spiral, $= \frac{s}{4r} \times \text{MC}^2$.

Also, since the triangles Mnr, MTC, are similar, MC : TC :: Mr : nr, or as r : s; wherefore

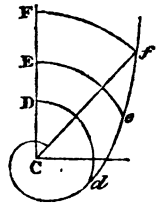
$$TC \times r = MC \times s, \text{ or } \frac{s \times MC}{4r} = \frac{1}{4}TC.$$

Whence the area of the whole spiral space

$$MDFC = \tfrac{1}{4}TC \times MC = \tfrac{1}{2} \text{ triangle } TCM.$$

OF THE HYPERBOLIC, OR RECIPROCAL SPIRAL.

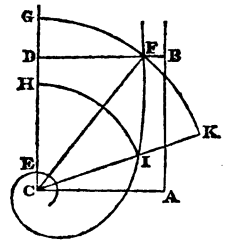
If from one of the extremities C, of the right line CF, as a centre, with the distances CD, CE, CF, &c. as radii, an indefinite number of arcs Dd, Ee, Ff, &c. be described, so as to be all equal to each other, the curve Cdef, drawn through their extremities, is called the hyperbolic, or reciprocal spiral (*l*).



The right line CF is called the axis of the spiral, and the point C its centre; also any right line Cf, drawn from the centre to the curve, is called an ordinate.

1. In the hyperbolic spiral EIF, any two ordinates CF, CI, are reciprocally to each other as the angles GCF, GCI, which they make with the axis CG.

(*l*) This curve appears to have received the above appellation, from the similarity of its equation with that of the common hyperbola, when referred to one of its asymptotes.



For let the circular arcs IH, FG, be described to meet the axis CG in H and G, and produce CI to meet the arc GFK in K.

Then, since CGK and CHI are similar sectors, we shall have CG : CH :: arc GK : arc HI.

But CG is equal to CF, and CH to CI, by construction; therefore, since the arc HI is also equal to the arc GF, CF : CI :: arc GK : arc GF.

Again, the arc GK : arc HI, or GF :: angle GCK, or GCI : angle GCF; whence CF : CI :: angle GCI : angle GCF.

2. Hence, every thing remaining as above, if the ordinate CF be put $=r$, the arc GF$=a$, the ordinate CI$=y$, and GK$=z$, we shall have

$$r : y :: z : a, \text{ or}$$
$$yz = ra.$$

Which expression is the equation of the spiral EIF.

3. If CA be drawn from the centre of the spiral, perpendicular to the axis CG, and made equal to either of the circular arcs HI, GF, &c. the right line AB, drawn through the point A, parallel to CG, will be an asymptote to the curve.

For through F, the extremity of the ordinate CF, draw the right line DFB parallel to CA, or perpendicular to CG.

Then, since DB is equal to CA (Euc. I, 34), and CA is equal to the arc GF (by construction), DB will also be equal to the arc GF.

But the arc GF, which is equal to the right line DB, being greater than its sine DF, the point B will fall without the curve EIF.

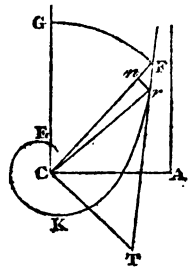
And if the ordinate CF be drawn at an indefinitely great distance from CA, with respect to the ordinate CI, the angle GCF will be indefinitely small with respect to HCI.

But when an angle is indefinitely small, the excess of the arc which measures it, above its sine, is less than any assignable magnitude.

Consequently, since FB is the excess of the arc GF above its sine DF, if the curve be continued till the angle GCF becomes indefinitely small, its distance FB, from AB, will be less than any assignable magnitude.

Wherefore, the right line AB is an asymptote to the curve EIF, as was to be shown.

4. If FT be a tangent to the curve at F, the area of the whole spiral space EKF, formed by an infinite number of revolutions, is equal to half the rectangle of the ordinate CF and the subtangent CT.



For draw the ordinate *cr* indefinitely near CF, and make *rn* perpendicular to CF.

Then since, by similar triangles, $Fn : nr :: FC : CT$, we shall have

$$CT \times Fn = FC \times nr.$$

But the triangle FCr is equal to $\frac{1}{2} FC \times nr$; wherefore, it is also equal to $\frac{1}{2} CT \times Fn$.

Hence, the sum of all the triangles FCr, formed by an infinite number of revolutions, is

$$= \tfrac{1}{2} CT \times \text{ all the } Fn\text{'s}.$$

And, therefore, since the sum of all the Fn's is $= FC$, the area of the whole spiral space EKF, which is composed of all the triangles FCr, is

$$= \tfrac{1}{2} CT \times FC.$$

Scholium. It might here, also, be proved, that the subtangent CT is equal to the arc GF, or to the distance CA between the centre and the asymptote; but this, and several other properties of the curves here treated of, cannot be easily demonstrated without the assistance of fluxions (*m*).

OF THE DESCRIPTION OF CURVES FROM THEIR EQUATIONS.

THIS branch of the subject, like that above given, is too extensive to be treated of at length in the present performance; but, if the following observations, with the examples subjoined to them, be properly attended to, the learner will not be at

(*m*) The reader, who may wish to see the investigations of other properties of these curves, may consult the small tracts that are given on these lines, by Emerson and Robertson, at the ends of their respective Treatises on the Conic Sections; of some particulars of which the author has availed himself in the present article; in which he hopes, however, the subject will be found to be treated in a more methodical and perspicuous manner than has usually been done.

a loss to perceive how any curve may be traced, and its limits and properties developed, in most cases of this kind that usually occur.

1. Supposing the abscissa of a curve to be represented by x, and the ordinate by y; then, if any particular value be assigned to x, in the equation denoting the nature of the curve, there will be as many values of y as there are units in the index of its highest power in that equation, and no more (*n*).

Thus, if $y^3 - a^2y + xy^2 - ax^2 = 0$ be the equation of some particular curve, and x be put $= b$, the equation, in this case, will become

$$y^3 - a^2y + by^2 - ab^2 = 0;$$

which has evidently three roots, or values of y, corresponding to the value of x that is denoted by b.

And by assigning any other value for x, as suppose c, there will arise the cubic equation

$$y^3 - a^2y + cy^2 - ac^2 = 0,$$

having three roots, or values of y, corresponding to the value of x denoted by c; and so on, for any other value of x.

And the same mode of reasoning will hold good for an equation of any order, or power, whatever.

2. Among the several values of the ordinate y, for any determinate value of x, those that are af-

(*n*) In order that the proposed equation may be the proper representative of some particular curve, it must, when irrational, be rendered free from surds; and, if there should be fractions in it, they must be made to disappear; which being done, the greatest number of dimensions, formed by the quantities x and y, will indicate the order to which the line belongs.

firmative must be drawn on the upper, or right hand side of the abscissa, and those that are negative on the contrary side.

3. Also, among the values of x, for any particular value of y, if those that are affirmative lie from the origin of the abscissa to the ordinate, on the right, those that are negative will lie from the origin of the abcissa, on the left.

4. When such a value of x is assigned, that some of the roots, in the resulting equation, are impossible, it is a sign that there are no ordinates answering to that supposition, as they cannot, in this case, reach the curve.

5. In assigning any value for x, if one root or value of y be 0, the curve will cut the abscissa in that point; but if y be infinite, the ordinate at that point will be an asymptote to the curve; which runs out *ad infinitum*.

6. Also, if x be infinite, and one value of y, in this case, becomes 0, the abscissa will be an asymptote to some branch of the curve, which runs out indefinitely, as before.

7. Hence, by assigning several values of x, we can determine the corresponding values of y, and, by that means, find as many points in the curve as may be judged necessary; and thus, by drawing the curve through them, obtain the *locus* of that equation.

8. Again, the equation of a curve, in order that it may have the greatest generality possible, ought to comprehend all the combinations of the co-ordinates x and y, which do not exceed the highest power of the equation.

Thus, for a line of the second order, for example, we ought to find y, y^2, xy, x, and x^2, in the equation, together with some known term; and each of the terms must have some indeterminate coefficient, which may be either positive, negative, or 0, in order that we may have all the varieties that the equation is susceptible of, both with regard to the signs of the terms, as well as to those that may be wanting.

9. Thus, the most general equation of the first degree is $ay+bx+c=0$, of the second degree $ay^2+bxy+cy^2+dx^2+ex+f=0$, of the third degree $ay^3+by^2+cxy^2+dx^2y+exy+fy+gx^3+hx^2+ix+k=0$, and so on.

In all of which cases, each term, except the first, which must be always affirmative, may be made either positive, negative, or 0, according as we take the values of the indeterminate coefficients a, b, c, &c.

But, it is more usual, for the sake of convenience, to arrange any equation of this kind, as, for example, that of the third degree, according to the order of the several powers of the ordinate, as follows: $ay^3+(bx+c)y^2+(dx^2+ex+f)y+(gx^3+hx^2+ix+k)=0$.

Where it is evident that $bx+c$ is the coefficient of the second term, dx^2+ex+f that of the third, and gx^3+hx^2+ix+k the fourth, or absolute term.

10. The equation may also be exhibited in the same manner, with respect to x and its powers, by putting y for the abscissa, and x for the ordinate; observing that the abscissa, in this case, must be taken from the same point as before, in a direction perpendicular to the former.

11. It may also be farther observed, that the same equation may give different curves according as the ordinates fall perpendicularly upon the axis, or obliquely.

Thus, the equation $y^2 = ax - x^2$ gives a circle, when the co-ordinates are perpendicular to each other; but when they are oblique, the curve, represented by that equation, will be an ellipse.

12. Hence, although the magnitude of the angle which the co-ordinates make with the axis, does not alter the generality of the equation, yet, in such cases, where a particular equation is given, the curve which it expresses can only be determined when the angle between the co-ordinates is known (*o*).

13. Another circumstance to be remarked in this doctrine is, that when the last, or constant term of the equation is wanting, it is a sign that the origin of the abscissa is taken from a point in the curve.

14. Also, if the point where this origin is taken belongs twice to the curve, which is the case with a double point, two roots of the equation, or values of the ordinate, will be equal to 0; in which case

(*o*) The chief, and almost general principle in the theory of curves, consists in the transformations of their equations; which are effected by transporting the origin of their co-ordinates, when necessary, to some other point that is more suitable to the proposed investigation; according to the manner pointed out in the first part of the present article: as also, by giving a different inclination to the principal axis and its ordinates; and then finding from the primitive equation, that of the curve resulting from these changes. But these, and various other particulars relating to this doctrine, are too numerous to be entered upon in the slight sketch of the subject here given.

the two last terms of the equation will be wanting, and *vice versa*.

15. If, therefore, in any equation of this kind, the constant term, and that which contains y, be wanting, it follows that the summit of the abscissa will be a double point; and if the terms which contain y^2, y, and y^0 (or the constant term), are not found in the equation, that point will be triple; and so on.

16. It is also to be observed, that the equation denoting the order of the curve, must not be decomposible into rational factors; as it would, in that case, comprehend several equations, each of which would belong to a particular curve, and their reunion would be all that the proposed equation could represent.

Thus, the equation $y^2 - ay - xy + ax = 0$, which is composed of the factors $(y - x) \times (y - a)$, comprises the two equations $y - x = 0$, and $y - a = 0$, each of which belong to the right line; the first forming with the axis, at the origin of the abscissæ, an angle of 45°, and the second will be parallel to the axis, at a distance $= a$.

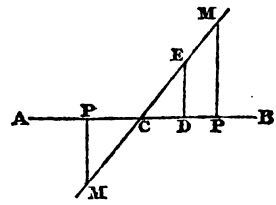
17. To this we may add, that curves of the 3d, 5th, 7th, or any odd number of dimensions, must have, at least, two arcs, or legs, proceeding towards contrary parts *ad infinitum;* since equations of these orders have always, at least, one real root; and, consequently, for every value of x in that equation, there must be a corresponding value of y; so that as x, or the abscissa, may be continually increased, on each side, it follows that there will be two branches of the curve, passing through the

extremities of y, that must go off indefinitely on both sides, or without limit.

18. But in curves of an even number of dimensions, as the roots of the equations by which they are denoted may be all impossible, it follows that a curve of this kind may be of the form of a circle, or oval, that is confined within certain bounds, beyond which it cannot extend.

19. The nature and number, indeed, of the branches of curves, form an important consideration in this doctrine; it being chiefly from these circumstances that certain analysts have established the first subdivision of them into orders and genera; but, for the reasons already given, we can here enter no farther into this part of the subject, than by illustrating it with the few following examples.

I. Let there be taken, in the first place, the equation $ay = bx + c$, which is that of the right line.



Then, if the abscissa x be taken on the line AB, from any point D, towards B, it is plain, that when $x = 0$, we shall have $y = \frac{c}{a}$.

Consequently, if DE be drawn at right angles to AB, and made equal to $\frac{c}{a}$, the line which belongs to

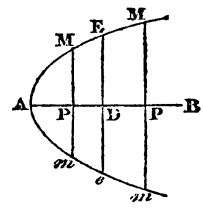
the proposed equation must pass through the point E.

Also, if the ordinate y be taken $=0$, we shall have $x=-\frac{c}{b}$; wherefore, if DC be taken on the contrary side of D, and made $=\frac{c}{b}$, the line, to which the equation belongs, must also pass through the point C.

Whence, as two points in the required line are now fixed, MCM must be the line answering to the proposed equation.

COR. If DP be made to represent any value of x, and the ordinate MP be drawn parallel to ED, MP will represent the corresponding value of y.

2. Let there be now taken the equation $y^2=ax+ab$, which is similar to that of the common parabola.



Then, supposing AB to be the line of the abscissæ, we shall have $y=\pm\sqrt{(ax+ab)}$, where a and b being given quantities, if DP, or x, be taken on the right from D, and assumed of some known value, $\sqrt{(ax+ab)}$ will also be known.

Whence, if PM and Pm be taken on contrary sides of AB, and be each made equal to $\sqrt{(ax+ab)}$, M and m will be two points in the required curve.

And, in like manner, it is plain, that for every

positive value of the abscissa x, taken in this way, we shall obtain a point of the locus of the curve on each side of AB.

It is also evident, that the greater DP, or x, is taken, the greater will $\sqrt{(ax+ab)}$ become; and, consequently, also, the greater will be PM and Pm.

If DP, or x, for instance, be supposed indefinitely great, PM and Pm will, likewise, become indefinitely great; and, therefore, the curve must go off to an infinite distance on each side of AB.

Also, if x be supposed to vanish, y will become $=\pm\sqrt{ab}$; so that the curve, in this case, will pass through E and e, and DE, De will be each a mean proportional between a and b.

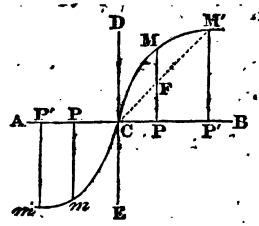
And if the point P be taken on the other side of D, x will become negative, and we shall have $y=\pm\sqrt{(ab-ax)}$; in which case it will have two values, as before, while x is less than b.

But if DA, or x, be taken $=b$, and the point P be supposed to come to A, ab will be $=ax$, and $y=0$; that is, PM and Pm will vanish, and the curve will cut AB in A.

Again, if the point P be taken on AB, produced to the left of A, x will become greater than b, and ax greater than ab; so that $\sqrt{(ab-ax)}$, or the two values of y, will, in that case, be imaginary.

Hence there are no ordinates to the left of A that will meet the curve; and, consequently, no part of the curve can lie on that side of A.

3. Next, let there be given the equation $y^3=a^2x$, which belongs to a line of the third order, to determine the curve.



For this purpose, draw the two indefinite right lines AB, DE, at right angles to each other; and suppose the values of x to be taken upon CB from the point C, and those of y upon CD, or upon lines parallel to it.

Then, from the nature of the proposed equation, it is plain, that when $x=0$, y will also be $=0$; and, consequently, the curve must pass through the point C.

In like manner, if $y=0$, it is obvious, that x will also be $=0$; and, therefore, the curve cannot cut the axis AB in any other point than C.

Wherefore, taking x, or CP $=\frac{1}{2}a$, we shall have $y=\sqrt[3]{\frac{1}{2}a^3}$, or $=\frac{1}{2}a\sqrt[3]{4}$; and, consequently, if PM be drawn parallel to CD, and made $=\frac{1}{2}a\sqrt[3]{4}$, M will be a point in the curve.

Also, by taking x, or CP′ $=a$, y will become $=\sqrt[3]{a^3}$, or a; and, therefore, drawing P′M′ parallel to CD, and making it $=a$, M′ will be another point in the curve.

And thus, by assuming other values of x, as many points in the curve may be found as shall be thought necessary.

Again, taking x indefinitely great, it is plain, from the given equation, that y will also be indefinitely great; and, consequently, the curve must pass on to infinity.

Also, since when $x=0$, y is $=0$, and when x is infinite y is infinite, it follows, that the curve can have no asymptotes that are parallel to the co-ordinates.

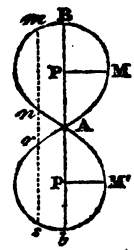
Farther, let the right line CM′ be drawn, cutting PM in F; then, because P′M′ is $=$CF′, PF will be $=$CP, or $\frac{1}{2}a$.

But PM, as above shown, is $=\frac{1}{2}a\sqrt[3]{4}$, which is manifestly greater than $\frac{1}{2}a$; wherefore, PM being greater than PF, the curve CMM′ must be concave to its axis CB.

Also, if x be taken negatively, or on the other side of the point C, we shall have $y=\sqrt[3]{-a^2x}$, or $-\sqrt[3]{a^2x}$; from which it is evident that there will arise as many negative and equal values of y, falling below AB, as there were positive values of it falling above AB.

Whence it follows, that the branch of the curve cmm' will be similar and equal to the branch CMM′, but having a contrary position.

4. Again, let the proposed equation be $a^2y^2=a^2x^2-x^4$, which belongs to a line of the fourth order, or to the curve called the *Lemniscate*.



Here, taking AB for the line of the abscissæ, and making it $=a$, we shall have $y=\pm\frac{x}{a}\sqrt{(a^2-x^2)}$;

which being $=0$ when $x=0$, it is plain that the curve will pass through the point A, from which the values of x are estimated.

Also, if x be taken on the upper side of A, and made $=a$, or on the under side and made $=-a$, y will be $=0$; and, consequently, the curve will pass through the points B, b, supposing Ab to be $=$ AB.

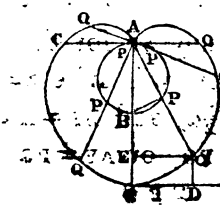
And if x be taken greater than a, it is plain that the two values of y will be imaginary; and, therefore, no part of the curve can lie beyond B or b.

Again, when $x=\pm\frac{1}{2}a$, or $\pm\frac{1}{2}$AB, we shall have $y=\pm\frac{1}{2}\sqrt{(a^2-\frac{1}{4}a^2)}$, or $=\pm\frac{1}{4}a\sqrt{3}$, and, therefore, if the ordinates PM, PM′, in this case, be made $=\frac{1}{4}a\sqrt{3}$, M and M′ will be two points in the curve.

And thus, by taking other values of x, and substituting them in the given equation, as many points in the curve may be found as shall be thought necessary.

From which process, repeated in the way here mentioned, it will appear that the curve consists of two equal and similar parts, having a double point at A; and that a right line ms may be drawn to cut it in four different points, agreeably to what has been before observed.

5. Next, suppose $y^4+(2x^2-ax-\frac{1}{4}a^2)y^2+x^4-ax^3=0$, or $a^2(ax^2+y^2)=4(x^2+y^2-\frac{1}{2}ax)^2$, to be the given equation; which also belongs to a line of the fourth order, or to the curve called the *Cardioide*.



Here, let $Aa(=a)$ be the line of the abscissæ, and the perpendicular aD, or some right line parallel to it, the line of the ordinates.

Then, it is evident, from the given equation, that when $y=0$, x will be $=a$; and, consequently, the curve will pass through the extremity a of the line Aa, and have AD a tangent to it at that point.

It is, also, equally manifest, that when $x=0$ y will be $=\frac{1}{2}a$; and, therefore, if AQ, or AC, which, in this case, are the values of y, be drawn perpendicular to Aa, and made $=\frac{1}{2}a$, Q and C will be two points in the curve.

And by finding, in like manner, other values of y, the locus of the curve, or any number of points in it, may be readily determined.

But its figure and construction will be best understood from the polar equation, which may be formed by putting AQ, or $\sqrt{(x^2+y^2)}=z$, and the angle $EAQ=\phi$, as usual.

In which case, we shall have $AQ(z)$: rad.(1) :: $AB(x)$: cos. ϕ, or $x=z$ cos. ϕ, and, consequently, by substitution, $a^2z^2=4(z^2-\frac{1}{2}az\cos.\phi)^2$.

Or, extracting the square root of each side of this last equation, and dividing by z,

$$a=2z-a\cos.\phi, \text{ or } z=\tfrac{1}{2}a+\tfrac{1}{2}a\cos.\phi.$$

Also, if Aa be bisected in B, and BP be drawn perpendicular to AQ, we shall have

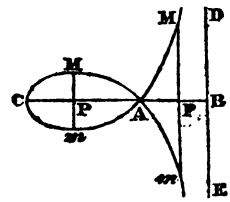
$AB(\frac{1}{2}a)$: rad.(1) :: AP : cos. ϕ; or $AP=\frac{1}{2}a\cos.\phi$.

Whence, substituting AP for this value of it in the last equation, there will arise $z=\frac{1}{2}a+AP$.

But $AQ=z$, $AB=\frac{1}{2}a$, and $AP=AQ-PQ$; wherefore, $AQ=AB+AQ-PQ$, or $AB=PQ$.

Hence, describing the circle APB, upon AB, or $\frac{1}{2}$Aa, as a diameter, and drawing APQ, on each side of Aa, so that PQ shall be every where = AB, the figure AQacA will be the required curve; having a cusp at A, and CQ or QQ = Aa or 2AB.

6. Lastly, let it be required to determine the locus, and other particulars, of the equation $ay^2 - xy^2 = x^3 + bx^2$, which represents a line of the third order.



For this purpose, take AB($= a$) for the line of the abscissæ, and the perpendicular BD, or a right line parallel to it, for the line of the ordinates, as usual.

Then, since, by the proposed equation, $y = \pm\left(\frac{x^3 + bx^2}{a - x}\right)^{\frac{1}{2}}$, it is plain that PM and Pm, taken on different sides of AB, must be equal to the several values of $\left(\frac{x^3 + bx^2}{a - x}\right)^{\frac{1}{2}}$, obtained by assuming certain particular values for x.

But if x, in the above expression, be taken = AB, or a, y will become infinite; and, consequently, DBE will be an asymptote to the curve.

And if x be supposed greater than a, or AP greater than AB, the fraction $\frac{x^3 + bx^2}{a - x}$, being, in that case, negative, its square root will be impossible, so that no part of the curve can fall beyond B.

It is also farther evident, that when $x=0$, y will be $=0$; from which it follows that the curve must pass through the point A.

Again, if x be supposed negative, or P be taken on the other side of A, we shall have, when x is less than b, $y=\pm(\frac{bx^2-x^3}{a+x})^{\frac{1}{2}}$; in which case, the two roots, or values of y, will be real and equal to each other.

But if $x=b$, the two roots, in that case, will both vanish, because $y=\pm(\frac{b^3-b^3}{a+b})^{\frac{1}{2}}=0$; and, therefore, if AC be taken $=b$, the curve will pass through the point C, forming a nodus between A and C.

And if x be taken greater than b, y, or its equal $\pm(\frac{bx^2-x^3}{a+x})^{\frac{1}{2}}$, will become imaginary; so that no part of the curve can fall beyond C.

Farther, if y be supposed $=0$, x^3+bx^2 will, also, $=0$; and as the three roots of this equation are $-b$, 0, and 0, it is obvious that the curve will have a double point at A, or must pass twice through it.

Also, if b be supposed to vanish, so that the equation becomes $ay^2-xy^2=x^3$, the nodus will likewise vanish, and the curve will consist of two arcs, forming a cusp at A; being, in this case, what is called the *cissoid of Diocles*, before treated of.

Having thus given a brief abstract of such parts of this doctrine as seemed best calculated to attract the attention of learners, I can here enter no farther into the subject, than barely to lay before the

reader a few particulars relating to the characters, properties, and species of lines of the third order, as treated of by Newton, in his celebrated treatise entitled *Enumeratio Linearum tertii Ordinis*, first published, at the end of his Optics, in 1706 (*p*).

Here, among other preliminary remarks, respecting the similarity of the parts and properties of these lines with those of the conic sections, the author observes, that these latter curves have diameters and axes, such that the right lines bisected by them are ordinates, and the intersection of the curve and diameter the vertex.

So, in lines of the third order, if any two parallel right lines, meeting the curve in three points, be cut by another right line, so as to make the sum of the two parts between the secant and the curve, on one side, equal to the third part, terminated by the curve on the other side, the same will be true of all other right lines parallel to these, that meet the curve in three points.

Hence, he observes, that these parts, so taken, may be called *ordinates*; the cutting line the

(*p*) A more acceptable present could scarcely be made to the British student, who has already acquired a certain portion of scientific knowledge, than a good translation of this work, and the Commentary on it, by Stirling, into the English language; with notes containing the observations and improvements of later writers: as there is no subject, perhaps, within the compass of pure mathematics, that can be considered as more beautiful, or better calculated to delight the mind, than the ingular and unexpected harmony of the properties of the various curves which arise out of this apparently complicated inquiry.

diameter; and when it meets the ordinates at right angles, the *axis;* the intersection of the diameter and curve, the *vertex;* the concourse of two diameters, the *centre;* and the concourse of all the diameters, the *common*, or *general* centre (*q*).

He next proceeds, after some farther remarks of the kind above mentioned, to observe, that the branches, or legs, of all lines of the second, third, and other higher orders, are either of the hyperbolic or parabolic kind; an hyperbolic leg being that which approaches indefinitely towards some asymptote, and a parabolic leg, that which has no asymptote.

These legs, he considers as best distinguished by their tangents; since, if the point of contact be at an infinite distance, the tangent of an hyperbolic leg will coincide with the asymptote, and that of a parabolic leg will recede indefinitely, or be no where to be found.

So that the asymptote of any leg of a curve will be found, by seeking the tangent of that leg at a point infinitely distant; and the direction, or course, of an infinite leg will be found, by seeking the position of a right line which is parallel to the

(*q*) From this it appears, that Newton considers the diameter of any curve, to be a right line, which divides two other parallel right lines in such a manner, that, in each of them, all the segments, or ordinates, on one side, between the diameter and different points of the curve, are equal to all those on the other side. But, according to others, a diameter is that line, whether right or curved, which bisects all the parallels drawn from one point of the curve to another.

tangent where the point of contact goes off *ad infinitum;* this line having, in that case, the same direction as the leg.

He then reduces all lines of the third order to four cases of equations, expressing the relations of their abscissæ and ordinates; and gives their enumeration, together with diagrams exhibiting all the forms of the several curves; the cases here mentioned being,

1st. $xy^2 + ey = ax^3 + bx^2 + cx + d$

2d. . . . $xy = ax^3 + bx^2 + cx + d$

3d. . . . $y^2 = ax^3 + bx^2 + cx + d$

4th. . . . $y = ax^3 + bx^2 + cx + d$

Where it is to be observed, that the coefficients a, b, c, d, e, denote given quantities, which may be taken either positively or negatively; and that any one, or more, of the terms of the equations may be wanting, provided the figure, by this means, does not become a conic section.

To this we may add, that under these four cases, considered in this way, a great variety of different forms of curves are found to arise, to which the author has given the following names:

1. An hyperbola lying wholly within the angle of the asymptotes, like the conic hyperbola, he calls an *inscribed hyperbola;* and that which cuts the asymptotes, and contains the parts cut off within its own periphery, a *circumscribed hyperbola.*

2. When the hyperbola has one of its infinite legs inscribed, or falling within one asymptote, and another circumscribed, or falling beyond the asymptote, it is called *ambigenial.*

3. That which has its legs looking towards each other, and directed the same way, is said to be *converging;* and that which has its legs looking contrary ways, *diverging*.

4. When the legs are convex different ways, it is called *cross-legged;* and when they cut the conjugate across, *cruciform.*

5. That which is applied to its asymptote, with a concave vertex and diverging legs, is called *conchoidal;* and that which cuts its asymptote with contrary flexures, and has its legs produced each way, is said to be *anguineal,* or *serpentine.*

6. When the curve returns round again, and cuts itself, it is called *nodated;* and if its parts concur, and terminate in the angle of contact, it is said to be *cuspidated.*

7. That whose conjugate has an oval, which vanishes, or becomes indefinitely small, is said to be *pointed,* or to have a *conjugate point;* and that which, on account of two of its roots being impossible, has neither an oval, cusp or point, is called *pure.*

8. Also, when the hyperbolic legs exceed in number those of the conic hyperbola, or are more than two, it is called a *redundant* hyperbola; and when it has fewer legs than the conic hyperbola, it is said to be a *defective* hyperbola.

9. He likewise calls such of these figures as have their ordinates determined by dividing the rectangle of the ordinate of a given conic section and a given right line by the common abscissa, *hyperbolismæ* of an hyperbola, ellipse, or parabola.

10. And in a similar manner, to what is above

shown of the hyperbola, he also denominates a parabola, *redundant, defective, diverging, converging, cruciform*, &c.

This being premised, it may now be observed, that the four cases of equations above mentioned furnish the following classes and varieties of these curves.

1. Nine redundant hyperbolas without diameters, having three asymptotes including a triangle; the first consisting of three hyperbolas, one inscribed, another circumscribed, and the third ambigenial with an oval; the second nodated; the third cuspidated; the fourth pointed; the fifth and sixth pure; the seventh and eighth cruciform; and the ninth anguineal.

2. Twelve redundant hyperbolas with only one diameter; the first oval; the second nodated; the third cuspidated; the fourth pointed; the fifth, sixth, seventh and eighth, pure; the ninth and tenth cruciform; and the eleventh and twelfth conchoidal.

3. Two redundant hyperbolas with three diameters, and nine others with three asymptotes, converging to a common point; the first being formed of the fifth and sixth redundant hyperbolas, whose asymptotes include a triangle; the second formed of the seventh and eighth; the third and fourth of the ninth; the fifth is formed of the fifth and seventh of the redundant hyperbolas with one diameter; the sixth of the sixth and seventh; the seventh of the eighth and ninth; the eighth of the tenth and eleventh; and the ninth of the twelfth and thirteenth; all of which conversions are effected by

diminishing the triangle comprehended between the asymptotes, till it vanishes into a point.

4. Six defective hyperbolas having no diameters; the first oval; the second nodated; the third cuspidated; the fourth pointed; and the fifth and sixth anguineal.

5. Seven defective hyperbolas having diameters; the first and second conchoidal with an oval; the third nodated; the fourth cuspidated (which is the cissoid of the ancients); the fifth and sixth pointed; and the seventh pure.

6. Seven parabolic hyperbolas having diameters; the first oval; the second nodated; the third cuspidated; the fourth pointed; the fifth pure; the sixth cruciform; and the seventh anguineal.

7. Four parabolic hyperbolas; four hyperbolisms of the hyperbola; three hyperbolisms of the ellipse; and two hyperbolisms of the parabola.

8. Six diverging parabolas; the first a trident; the second oval; the third nodated; the fourth pointed; the fifth cuspidated (which is Neil's parabola); and the sixth pure.

9. Lastly, one, commonly called the cubical parabola.

In which cases it may be observed, that, among the number here mentioned, sixty-five are derived from the first of the four equations above given; one from the second; five from the third; and one from the fourth; making in all, according to the enumeration of Newton, seventy-two different species.

But Stirling has shown this enumeration to be imperfect, having added to the number four more redundant hyperbolas; and Nicholas Bernoulli,